Math 104
Topology

The classification of surfaces

There's something oddly appealing about starting off the topology seminar with a theorem in topology. Thus we begin with a basic, but nontrivial, result—the classification of surfaces—despite the fact that, having had zero class meetings so far, it may seem that we are in no position to understand the details of the proof or even what the theorem says. After a couple of months, we'll run through this again and everything will make perfect sense. (That may have been a joke, but I hope not.) Whether it's out of place or not, the theorem is attractive as a way to introduce topology, because the arguments make intuitive geometric sense and, most importantly, they convey the spirit of how topology works (roughly speaking, anything goes).

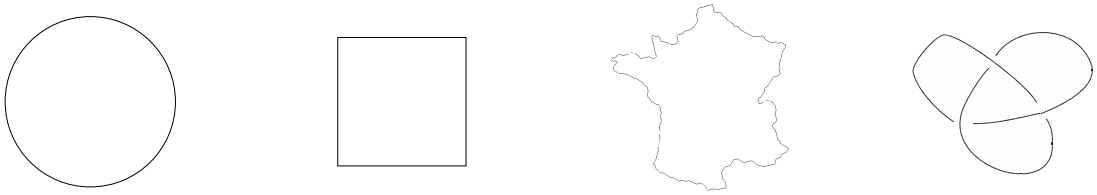
The upshot of the theorem is that it's possible to completely describe all compact surfaces. That is, the theorem provides a list of surfaces such that every possible surface must be "the same" as one of those on the list. Moreover, no two on the list are "the same." Indeed, by the end of the semester, we shall see how to convert this essentially geometric problem into algebraic or combinatorial computations that determine whether two given surfaces are the same.

Two questions arise naturally:

(1) What is a surface?
(2) What does it mean for two of them to be "the same"?

Well, first of all, a surface is a special type of *topological space*. For the moment, it's OK to think of a topological space as a fancy name for a subset of $\mathbf{R}^n$. Even better: think of it as another name for metric space. In due course, we will learn the correct definition. Whatever it is, a topological space may or may not possess a variety of properties. Among these are two with which you are familiar from real analysis, namely, *compactness* (every open cover has a finite subcover) and *connectedness* (there is no partition into two nonempty, disjoint open subsets).

Two spaces X and Y are called *homeomorphic* if there exists a bijective, continuous function $f\colon X \to Y$ whose inverse $f^{-1}\colon Y \to X$ is also continuous. Such an f (or f^{-1}) is called a *homeomorphism*. For instance, all the curves in Figure 1 are homeomorphic to one another. It's pretty easy to imagine how to define continuous bijections between any two of them.



circle square outline of France trefoil knot

Figure 1. A circle and some curves that are homeomorphic to it.

Homeomorphism is the topological version of isomorphism. Thus one should regard two homeomorphic spaces as being essentially the same—they are just continuous distortions of one another.

Now on to surfaces: On an intuitive level, one probably thinks of a surface as some sort of two-dimensional type of object (as opposed to a curve, which is one-dimensional, or a solid, which is three-dimensional). The most obvious example of a surface is a plane in $\mathbf{R}^3$. In general, unlike planes, most surfaces fail to be completely flat, but at least small pieces of them will resemble (i.e., be homeomorphic to) small pieces of a plane. With this as motivation, let us define a subset S of $\mathbf{R}^n$ to be a *surface* if every point x in S is contained in a subset U of S which is (relatively) open in S and homeomorphic to the open disk $D^2 = \{\, a \in \mathbf{R}^2 \,:\, \|a\| < 1 \,\}$ in $\mathbf{R}^2$. For instance, the unit sphere $S^2 = \{\, x \in \mathbf{R}^3 \,:\, \|x\| = 1 \,\}$ is a surface. A typical x and U are shown in Figure 2.

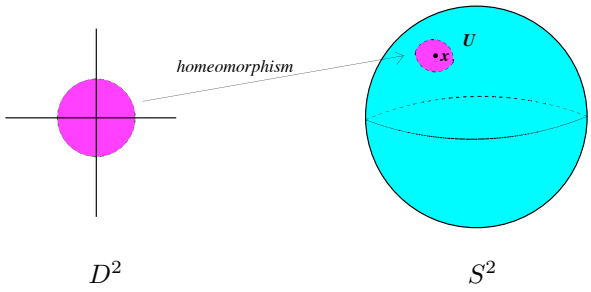


Figure 2. Small pieces of a sphere are homeomorphic to small pieces of a plane.

And here's the ever-popular Parade of Doughnuts*:

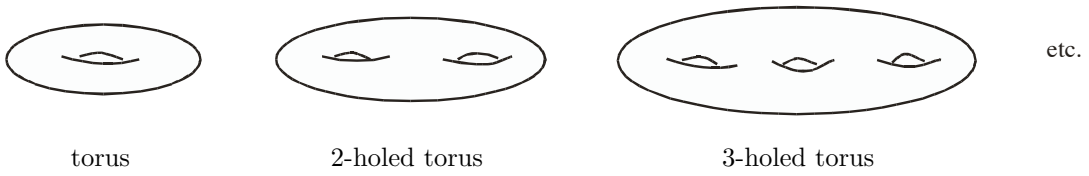


Figure 3. The Parade of Doughnuts.

There is a simple binary operation that one can perform on surfaces in the following way. Given two surfaces S_1 and S_2, define their *connected sum* $S_1 \# S_2$ to be the surface obtained by first cutting out and then discarding a small disk from each surface and then gluing the two remaining pieces together along their boundary circles. For example, the connected sum of two tori is a 2-holed torus, as indicated in Figure 4.

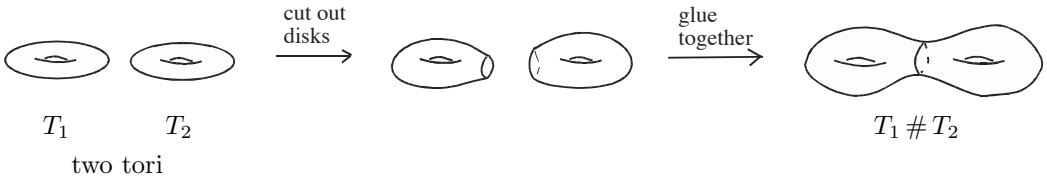


Figure 4. The connected sum of two tori is a 2-holed torus.

A general formula: (m-holed torus) # (n-holed torus) = (($m+n$)-holed torus).

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Let's get down to business and confront the problem of trying to describe all possible compact surfaces. Apart from doughnuts and the sphere, what else could there possibly be? What else could one want? (Note: pizza = sphere.) One complication is that there may be subsets of some high-dimensional $\mathbf{R}^n$ that satisfy the definition of surface. We cannot hope to picture what such surfaces might look like accurately, but fortunately there is an easy combinatorial way of constructing surfaces in the abstract that allows us to imagine further examples.

The idea is to take a polygon and glue together the edges in certain designated pairs. This process is called forming a "quotient" of the polygon. We'll discuss this in detail later on in the semester. For instance, the torus can be obtained by gluing together opposite sides of a square, as shown in Figure 5. The notation

* The drawings in the figure should be shaded in to indicate that they are surfaces, rather than collections of curves, and it is only to simplify the graphics that they are not. A similar comment applies to the other unshaded figures in the rest of the handout.

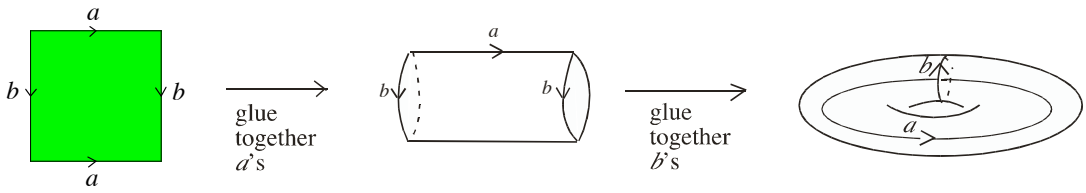

Figure 5. The torus as a quotient of a square, $S_{aba^{-1}b^{-1}}$.

for such a polygon is to start at some edge, say the a at the top, and then read around clockwise, taking into account the orientation of the edges. Thus the square in Figure 5 would be denoted $aba^{-1}b^{-1}$. An expression like "$w = aba^{-1}b^{-1}$" is called a *surface symbol*. We denote the surface to which it leads by S_w.

Now let's alter this example slightly and consider the surface symbol $w = aba^{-1}b$. This time, S_w is called the *Klein bottle*; an attempt to draw it is made in Figure 6.

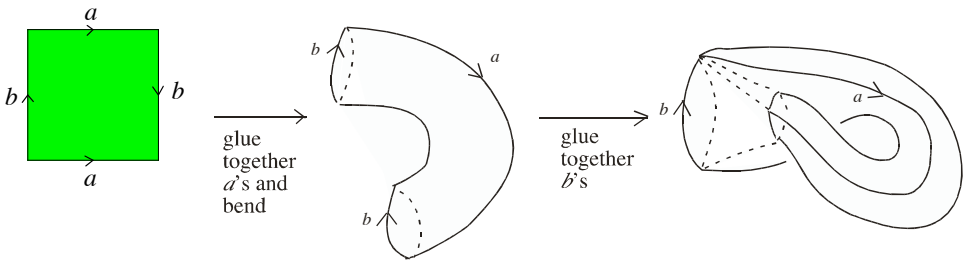

Figure 6. The Klein bottle, $S_{aba^{-1}b}$.

This picture of the Klein bottle in $\mathbf{R}^3$ is not really right. In fact, the object drawn is not even a surface. The real Klein bottle does not cut through itself in a circle of self-intersection the way we forced it to do above. It turns out that it's impossible to draw a correct picture in 3-dimensional space: if you really want to see a Klein bottle, you'll have to go to $\mathbf{R}^4$. (Keep in mind, however, that going to $\mathbf{R}^4$ is not a legitimate excuse for missing a meeting of the seminar.)

Two further examples are fundamental. First, the sphere S^2 can be obtained through the polygon method by starting with $w = aa^{-1}$.

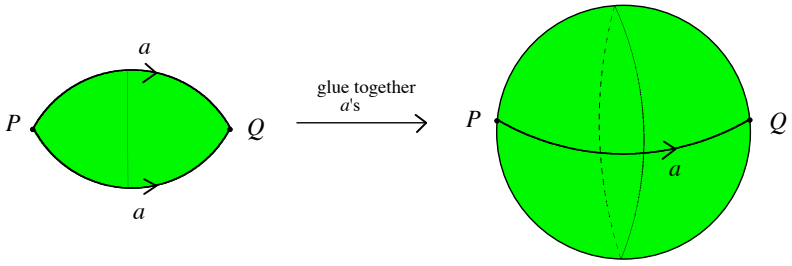

Figure 7. The sphere as a quotient of a 2-gon: $S^2 = S_{aa^{-1}}$.

And, lastly, consider the surface corresponding to $w = aa$ in Figure 8.

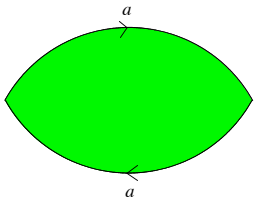

Figure 8. S_{aa}, the projective plane.

This surface is called the *projective plane*. Like the Klein bottle, the projective plane can be seen in $\mathbf{R}^4$, but not in $\mathbf{R}^3$. Unlike the Klein bottle, it takes some ingenuity even to draw an incorrect picture of it in $\mathbf{R}^3$. Just to get some feel for what it's like, let's cut it apart and put it back together, as in Figure 9. The piece labeled "M" is a *Möbius strip*. Note that it has one big edge (homeomorphic to a circle) going all the way around its boundary. The other piece is a closed disk. It too has boundary homeomorphic to a circle. Thus these pictures show that, if you were to glue a disk onto a Möbius strip by attaching them along their boundaries, you'd end up with a projective plane.

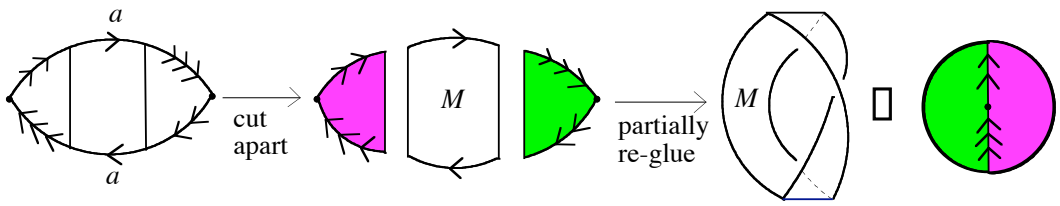

Figure 9. Projective plane = Möbius strip ∪ disk.

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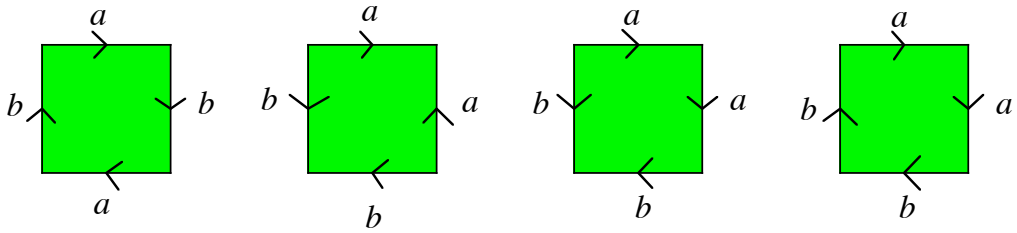
Now, it's possible to draw lots of polygons and glue together pairs of their edges in many different ways. For example, Figure 10 illustrates some edge identifications for a square different from those that produced the torus and Klein bottle earlier.



(i) $abab$ (ii) $aa^{-1}bb^{-1}$ (iii) $aabb^{-1}$ (iv) $aabb$

Figure 10. More quotients of a square.

As one moves on to quotients of hexagons, octagons, etc., the possibilities multiply further. As a result, one might despair of ever keeping track of all the surfaces that could result. This would be a waste of perfectly good despair, however, for it turns out that there is actually quite a bit of structure in the process. For instance, each of the quotients in Figure 10 duplicates a surface we have already encountered, a fact which you are asked to verify in the exercises for the first week. There is similar redundancy for polygons with a greater number of sides. In other words, even though there is a multitude of different polygons, they don't all lead to different surfaces. This brings us to the main theorem.

Theorem. (Classification of surfaces) *Every compact, connected surface is homeomorphic to one and only one of the following:*

(a) the sphere S^2;
(b) a connected sum of n tori ($n \geq 1$); or
(c) a connected sum of n projective planes ($n \geq 1$).

As a not-entirely-obvious example, we can deduce that the Klein bottle must be homeomorphic to one of the surfaces listed in the theorem. In fact, it is a connected sum of two projective planes, as illustrated by the sequence of cutting and pasting operations diagrammed in Figure 11.

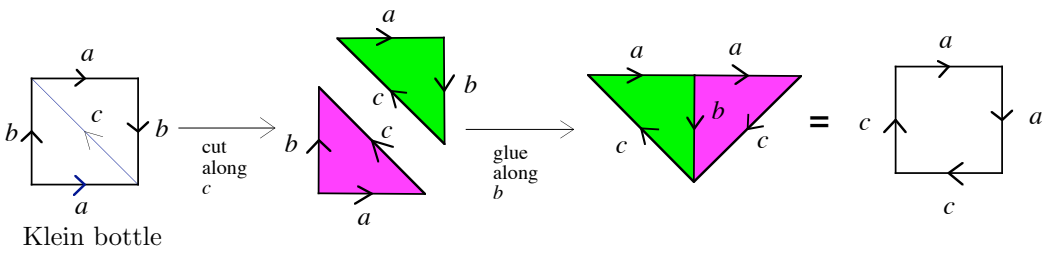


Figure 11. Klein bottle = connected sum of two projective planes.

That the final polygon really does represent a connected sum of two projective planes follows from Figure 1.9 in the book *Algebraic Topology: An Introduction*, by William S. Massey, Chapter 1 of which is the remaining part of the first reading assignment. Here is a brief summary of Massey's arguments.

I. Every surface S is homeomorphic to S_w for some surface symbol w.

(Idea of proof: Cut S up into triangles and lay these triangles out in the plane. Then glue as many of them back together as possible while remaining in the plane. The end result will be a polygon whose edges still need to be glued together in order to recover the original S. The way these final edges should be glued can be encoded by a surface symbol w that will make $S = S_w$.)

II. Every S_w is homeomorphic to S_v, where v is a surface symbol of one of the following types:

(a) $v = aa^{-1}$ (sphere)
(b) $v = a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n{}^{-1}b_n{}^{-1}$ (connected sum of n tori)
(c) $v = a_1a_1a_2a_2\cdots a_na_n$ (connected sum of n projective planes).

(Idea: Cut and paste. In fact, Massey gives an algorithm for converting S_w to S_v.)

III. No two of the S_v's in part II are homeomorphic.

(Idea: Orientability and the Euler characteristic.) □

You should read through Massey's proof carefully, but don't worry about terminology you haven't seen before. In particular, here is a list of words *not* to worry about:

(1) topological spaces (just equate them with subsets of $\mathbf{R}^n$);
(2) Hausdorff space or T_2 separation axiom;
(3) locally compact;
(4) second axiom of countability or countable base;
(5) quotient space;
(6) closed map.

The most important topics of the reading are in Sections 3, 4, 5, and *especially* 7 and 8.

As Massey observes, the classification theorem is unusually satisfying in that it gives such a complete solution. Go ahead and feel good about it, but don't expect it to happen again.