Ben Good
week 11

WARNING: Not Readable!

* we first want our solution to be stable when everything is well mixed, i.e. $\nabla^2 s = 0$

so we want:

$$\begin{aligned} s_t &= \gamma g \\ a_t &= \gamma f \end{aligned} \quad \text{to be stable.}$$

* thus, we linearize

$$J = \gamma \begin{pmatrix} g_s & g_a \\ f_s & f_a \end{pmatrix}, \quad \text{solutions are } e^{\lambda t}$$

where

$$|\gamma J - \lambda I| = 0 \quad \text{or} \quad \begin{vmatrix} \gamma g_s - \lambda & \gamma g_a \\ \gamma f_s & \gamma f_a - \lambda \end{vmatrix} = 0$$

or

$$(\gamma g_s - \lambda)(\gamma f_a - \lambda) - \gamma^2 f_s f_a = 0$$

$$\lambda^2 - \gamma(g_s + f_a) + \gamma^2(g_s f_a - f_s g_a) = 0$$

so if λ is always negative,

$$-\gamma(g_s + f_a) > 0 \quad \text{or} \quad \boxed{g_s + f_a < 0}$$

$$\gamma^2(g_s f_a - f_s g_a) > 0 \quad \text{or} \quad \boxed{g_s f_a - f_s g_a > 0}$$

Now we look at full problem near homogeneus eq.

$$s = s^* + s' \quad , \qquad w = \begin{bmatrix} s' \\ a' \end{bmatrix}$$

so guess $\quad w = e^{\lambda t} W_k(\vec{r})$

which works if

$$\lambda w = \gamma J w - D k^2 w \quad , \quad D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$$

or $\quad \left| \gamma J - D k^2 - \lambda I \right| = 0$

or
$$\begin{vmatrix} (\gamma g_s - k^2) - \lambda & \gamma g_a \\ \gamma f_s & (\gamma f_a - d k^2) - \lambda \end{vmatrix} = 0$$

so conditions are either

$$(\gamma g_s - k^2) + (\gamma f_a - d k^2) > 0$$

$$\gamma (g_s + f_a) - (1 + d) k^2 > 0 \rightarrow \text{but always negative}$$

or
$$(\gamma g_s - k^2)(\gamma f_a - d k^2) - \gamma^2 f_s g_a < 0$$

or

$$dk^4 - \gamma(dg_s + f_a)k^2 + \gamma^2(g_s f_a - f_s g_a) < 0$$

so $dk^4 > 0$, last term > 0,

so at minimum,

$dg_s + f_a > 0$, but $g_s + f_a < 0$,

so $d \neq 1$, g_s, f_a diff't. signs.

Final condition:

$$dg_s + f_a > 2\sqrt{d(g_s f_a - f_s g_a)} > 0$$

so all conditions for diffusive instability:

$$\boxed{\begin{array}{l} g_s + f_a < 0 \\ g_s f_a - f_s g_a > 0 \\ d \neq 1,\ f_a, g_s \text{ diff't. signs.} \\ dg_s + f_a > 2\sqrt{d(g_s f_a - f_s g_a)} > 0 \end{array}}$$

In 1 dimension,

$$\nabla^2 w_k = \frac{\partial^2 w_k}{\partial x^2}, \quad \text{so} \quad w_k = A\sin kx + B\cos kx$$

and boundary conditions imply

$$A = 0, \quad k = \frac{\pi n}{L}$$

In higher dimensions, tougher

thus, we substitute in w, and get

$$\lambda w = \gamma J w - Dk^2 w$$

so w is a solution iff

$$|\lambda - \gamma J + Dk^2| = 0$$

$$\begin{vmatrix} \lambda - \gamma g_s + k^2 & -\gamma g_a \\ -\gamma f_s & \lambda - \gamma f_a + dk^2 \end{vmatrix} = 0$$

But now we want $\lambda > 0$ for some k^2 so that our disturbances can form patterns.

by multiplying out, we obtain

$$-\gamma^2 g_a f_s + \lambda^2 + \gamma^2 g_s f_a + dk^4 - \gamma\lambda(f_a + g_s) + \lambda(k^2 + dk^2)$$
$$- \gamma g_s d k^2 - \gamma f_a k^2$$

$$\boxed{\begin{aligned} &\lambda^2 + \lambda\left[k^2(1+d) - \gamma(f_a + g_s)\right] + h(k^2) = 0 \\ &h(k^2) = dk^4 - \gamma(dg_s + f_a)k^2 + \gamma^2(g_s f_a - g_a f_s) \end{aligned}}$$

so if λ is to be negative, either

$$\left[k^2(1+d) - \gamma(f_a + g_s)\right] > 0$$ * different from book

or

$$h(k^2) < 0$$

but $(g_s f_a - g_a f_s)$ is > 0, so we must have

$dg_s + f_a > 0$. But $g_s + f_a < 0$, so we must have

$d \neq 1$ and g_s, f_a have opposite signs.

so first we want to make sure the system is stable without diffusion

so we want

$$s_t = \gamma g(s,a) \quad \text{to be stable}$$
$$a_t = \gamma f(s,a)$$

what do we do? $\boxed{\text{linear analysis.}}$

$$J = \begin{pmatrix} \gamma g_s & \gamma g_a \\ \gamma f_s & \gamma f_a \end{pmatrix}$$

solutions $e^{\lambda t}$ where λ is an eigenvalue

i.e.

$$(\cancel{\gamma} g_s - \lambda)(f_a - \lambda) - \cancel{\gamma} f_s g_a = 0$$

or $\lambda^2 - \lambda g_s - \lambda f_a - f_s g_a + g_s f_a = 0$

or $\lambda^2 - (g_s + f_a)\lambda - f_s g_a = 0$

$A = 1$, $B = -(g_s + f_a)$ $C = -f_s g_a + g_s f_a$

so $$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2}$$

Now, we see that we must have $\lambda < 0$ for stability, so

i. $\boxed{g_s + f_a < 0}$ and also

If $-4AC$ is positive $\sqrt{\quad''} > B$, so λ positive.

Thus, $\boxed{-4AC = \text{negative}}$

so $C =$ positive, $g_s f_a - f_s g_a > 0$

thus

$$\boxed{\begin{array}{l} \text{conditions for stability w/o diffusion:} \\ g_s + f_a < 0 \\ g_s f_a - f_s g_a > 0 \end{array}}$$

Now we look at what happens with diffusion:

still around steady state:

guess $w_t = e^{\lambda t} W_k(\vec{r})$, where W_k satisfies:

$$\nabla^2 W_k + k^2 W_k = 0$$

14.6

If the population is to die out,

$u^* = 0$ must be an attractor

$u^* = \dfrac{(r-E)K}{r}$ must be unstable

we then linearize the D.E. around u^*:

$$u'_t = f'(u^*)u' + Du'_{xx}, \quad u = u' + u^*$$

$$= \left((r-E) - \frac{2ru^*}{K}\right)u' + Du'_{xx}$$

we guess a solution of the form

$$u' = e^{\lambda t}(A\sin kx + B\cos kx)$$

so the boundary conditions yield:

$$u'_x(0,t) = e^{\lambda t}k(A\cos 0 - B\sin 0) = e^{\lambda t}kA = 0$$

or $A = 0$

14.6 Consider the following system

$$u_t = ru\left(1-\frac{u}{K}\right) - Eu + Du_{xx}$$

with

$$u_x(0,t) = 0. \quad u(H,t) = 0$$

and $\quad r > 0, \; r > E > 0, \; D > 0$

The homogeneous steady state is given by

$$0 = ru^*\left(1-\frac{u^*}{K}\right) - Eu^*$$

$$0 = (r-E)u^* - \frac{ru^{*2}}{K}$$

$$0 = \left((r-E) - \frac{ru^*}{K}\right)u^*$$

so either

$$u^* = 0 \quad \text{or} \quad u^* = \frac{(r-E)r}{K}$$

14.6

and

$$u'(H,t) = e^{\lambda t} B \cos kH = 0 \quad \text{or} \quad kH = \frac{n\pi}{2}, \text{ n odd}$$

or $k = \frac{n\pi}{H2}$.

Then upon substituting back, we obtain

$$\lambda = \left((r-E) - \frac{2ru^*}{K} - D\left(\frac{n\pi}{2H}\right)^2 \right)$$

So for $u^* = 0$

$$\lambda = (r-E) - \frac{D(n\pi)^2}{4H^2} \quad \text{and} \quad \lambda < 0 \text{ when}$$

$$H < \left(\frac{D}{r-E}\right)^{1/2} \frac{\pi}{2}$$ so this is one necessary condition for the population to die out

For $u^* = \left(\frac{r-E}{r}\right)K$,

$$\lambda = -(r-E) - \frac{D(n\pi)^2}{4H^2}$$ $\lambda > 0$ when never?

14.7

Consider the spruce budworm model:

$$u_t = ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2} + Du_{xx}\,, \quad u=0 \text{ on } x=0, L$$

we let $x \to x/L$, and look for U s.t. $\frac{dU}{dt}=0$

so $$f(U) + \frac{D}{L^2}u_{xx} = 0 \quad \text{or} \quad L^2 f(u) + Du_{xx} = 0$$

we seek an L such that $\exists$ 3 distinct roots.

so we note that $U \approx A\sin(\pi x)$
and we obtain

$$L^2 f(U) - D\pi^2 U \approx 0$$

or $$\frac{D\pi^2 U}{L^2} \approx f(U) = ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2}$$

14.7

$$f'(u) = r - \frac{2ru}{q} - \frac{(1+u^2)2u - u^2(2u)}{(1+u^2)^2}$$

$$= r - \frac{2ru}{q} - \frac{2u}{(1+u^2)^2}$$

So $f'(u) = \frac{D\pi^2}{L^2}$ and $\frac{D\pi^2 u}{L^2} = f(u)$

at L_0 or when

$$\frac{D^2\pi^2}{L^2} = r - \frac{2ru}{q} - \frac{2u}{(1+u^2)^2}$$

$$\frac{D^2\pi^2}{L^2} = r\left(1 - \frac{u}{q}\right) - \frac{u}{(1+u^2)}$$

or $$\frac{ru}{q} + \frac{2u}{(1+u^2)^2} - \frac{u}{(1+u^2)} = 0$$

or $$\frac{r}{q}(1+u^2)^2 - (1+u^2) + 2 = 0 \Rightarrow (1+u^2)^2 - \frac{q}{r}(1+u^2) + \frac{2q}{r} = 0$$

$$u^4 + \left(\frac{2r}{q} - 1\right)u^2 + 2 + \frac{r}{q} = 0$$

11.21

we have the reaction diffusion system:

$S(x,t)$ = concentration of substrate

$A(x,t)$ = concentration of cosubstrate

In the active layer, S and A react at the rate

$$R = \frac{V_m A S}{K_m + S + S^2/K_s}$$

and S and A are fed by a reservoir at concentration S_0 and A_0 and diffuse, so

$$S_t = \frac{D_s'}{L_1 L_2}(S_0 - S) - R + D_s \nabla^2 S$$

$$A_t = \frac{D_A'}{L_1 L_2}(A_0 - A) - R + D_A \nabla^2 S$$

if we define $s = \frac{S}{K_m}$, $a = \frac{A}{K_m}$, $t^* = \frac{t D_s}{L^2}$, $\nabla^{*2} = L^2 \nabla^2$

11.21

then we get $S = K_m s$, $A = a k_m$,

$$t = \frac{L^2 t^*}{D_s} \Rightarrow dt = \frac{L^2}{D_s} dt^*, \quad \nabla^2 = \frac{\nabla^*}{L^2}$$

$$S_t = \frac{D_s K_m}{L^2} \frac{ds}{dt^*}, \quad A_t = \frac{D_s K_m}{L^2} \frac{da}{dt^*}$$

$$R = \frac{V_m K_m^2 sa}{K_m + K_m s + K_m^2 s^2 / K_s} = \frac{V_m K_m sa}{1 + s + \frac{K_m}{K_s} s^2}$$

$$s_t = \frac{L^2}{D_s K_m} \left[\frac{D_s' K_m (s_0 - s)}{L_1 L_2} - \frac{V_m K_m sa}{1 + s + \frac{K_m}{K_s} s^2} + \frac{K_m D_s}{L^2} \nabla^2 s \right]$$

$$a_t = \frac{L^2}{D_s K_m} \left[\frac{D_a' K_m (a_0 - a)}{L_1 L_2} - \frac{V_m K_m sa}{1 + s + \frac{K_m}{K_s} s^2} + \frac{K_m D_a}{L^2} \nabla^2 a \right]$$

11.21

$$s_t = \frac{L^2}{L_1 L_2} \frac{D_s'}{D_s}(s_0 - s) - \frac{L^2 V_m}{D_s} F(s,a) + \nabla s$$

$$a_t = \frac{L^2}{L_1 L_2} \frac{D_a'}{D_s}(a_0 - a) - \frac{L^2 V_m}{D_s} F(s,a) + \frac{D_a}{D_s} \nabla^2 a$$

where $F(s,a) = \dfrac{sa}{1+s+Ks^2}$, $K = \dfrac{K_m}{K_s}$

or

$$s_t = \frac{L^2 D_s'}{L_1 L_2 D_s}\left[s_0 - s - \frac{L_1 L_2 V_m}{D_s'} F(s,a) \right] + \nabla^2 s$$

$$a_t = \frac{L^2 D_s'}{L_1 L_2 D_s}\left[\frac{D_a'}{D_s'}(a_0 - a) - \frac{L_1 L_2 V_m}{D_s'} F(s,a) \right] + \frac{D_a}{D_s} \nabla^2 a$$

and if $\gamma = \dfrac{L^2 D_s'}{L_1 L_2 D_s}$, $\alpha = \dfrac{D_a'}{D_s'}$, $\beta = \dfrac{D_a}{D_s}$

$$\rho = \frac{L_1 L_2 V_m}{D_s'}$$

11.21

then we obtain

$$s_t = \gamma\, g(s,a) + \nabla^2 s\,, \quad a_t = \gamma\, f(s,a) + \beta \nabla^2 a$$

where

$$g(s,a) = (s_0 - s) - \rho F(s,a)$$

$$f(s,a) = \alpha(a_0 - a) - \rho F(s,a)$$

$$F(s,a) = \frac{sa}{1+s+Ks^2}$$

and

$$K = \frac{K_m}{K_s}\,, \quad \alpha = \frac{D_a'}{D_s'}\,, \quad \beta = \frac{D_a}{D_s}\,, \quad \rho = \frac{L_1 L_2 V_m}{D_s'}$$

$$\gamma = \frac{L^2 D_s'}{L_1 L_2 D_s} = \frac{D_s'}{L_1 L_2} \Big/ \frac{D_s}{L^2}$$

11.21

a homogeneous steady state $(\bar{s}, \bar{a})$

has $\quad g(\bar{s},\bar{a}) = f(\bar{s},\bar{a}) = 0$

i.) $0 = (s_0 - s) - \rho F(s,a)$

ii) $0 = \alpha(a_0 - a) - \rho F(s,a)$

$$\frac{\rho s a}{1+s+ks^2} = s_0 - s \Rightarrow a = \frac{s_0 - s}{\rho s}(1+s+ks^2)$$

(from i.)

so from ii.) $\qquad$ cubic?

$$0 = \alpha a_0 - \alpha a - \frac{\rho s a}{1+s+ks^2}$$

$$0 = \alpha a_0 - \alpha \frac{(s_0 - s)}{\rho s}(1+s+ks^2) - (s_0 - s)$$

$$0 = \rho \alpha a_0 s - \alpha(s_0 - s)(1+s+ks^2) - \rho s(s_0 - s)$$

11.21

the Jacobian is given by

$$\hat{J} = \begin{pmatrix} g_s & g_a \\ f_s & f_a \end{pmatrix}$$

where

$$g_s = -1 - \rho F_s = -1 + \rho \left[\frac{(1+s+ks^2)a - sa(1+2ks)}{[1+s+ks^2]^2} \right]$$

$$= -\rho \frac{a - ks^2 a}{(1+s+ks^2)^2} - 1$$

$$g_a = 0 - \frac{\rho s}{1+s+ks^2} < 0$$

$$f_s = 0 - \rho \frac{a - ks^2 a}{(1+s+ks^2)^2}$$

$$f_a = -\alpha - \frac{\rho s}{1+s+ks^2} < 0$$

$$\begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

activator
inhibitor

11.21

i.) Clearly, $g_a, f_a < 0$ because $F_a, \alpha, \rho > 0$

Our conditions for instability are then

$$g_s + f_a < 0, \quad g_s f_a - g_a f_s > 0$$

at the steady state

Since $g_a, f_a < 0$, $|g_a|, |f_a| > 0$ and

we have

$$-g_s |f_a| + |g_a| f_s > 0 \quad \text{or}$$

$$|g_a| f_s > g_s |f_a| \quad \text{or} \quad \frac{|g_a|}{|f_a|} f_s > g_s$$

If $f_s > 0$, we have

$$\frac{|g_a|}{|f_a|} > \frac{g_s}{f_s} \Rightarrow \frac{\rho F_a}{\alpha + \rho F_a} > \frac{-1-\rho F_s}{-\rho F_s} = 1 + \frac{1}{\rho F_s}$$

But $F_a > 0$, so LHS is always less than 1 so we have

$$1 > 1 + \frac{1}{\rho F_s} \quad \text{or} \quad 0 > \frac{1}{\rho F_s} \quad \text{or} \quad F_s < 0$$