we first want our solution to be stable when everything is well mixed, i.e. $\nabla^2 s = 0$ so we want:

$$s_t = \delta g$$
$$a_t = \delta f$$

thus, we linearize

$$J = \gamma \begin{pmatrix} g_s & g_a \\ s_s & s_a \end{pmatrix}$$

solutions are $e^{At}$

where

$$|\delta J - x I| = 0 \text{ or } \begin{vmatrix} \delta g_s - \lambda & \delta g_a \\ \delta s_s & \delta s_a - \lambda \end{vmatrix} = 0$$

or

$$(\delta g_s - \lambda)(\delta s_a - \lambda) - \delta^2 s_s s_a = 0$$

$$(g_s + s_a)^2 + \gamma^2 (g_a f_a - s_g g_a) = 0$$

so if $\lambda$ is always negative,

$$-\delta (g_s + 5a) > 0 \text{ or } g_s + 5a < 0$$

$$\gamma^2 (g_s f_a - s_g g_a) > 0 \text{ or } g_s f_a - s_g g_a > 0$$
Now we look at full problem near homogeneous eq.

$$s_0 = s^* + s^1; \quad \mathbf{w} = \begin{bmatrix} s^1 \\ a^1 \end{bmatrix}$$

so guess \( w = e^{Ate^{T}} \)

which works if

$$\lambda w = \gamma J w - Dk^2 w, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$$

or

$$\gamma J - Dk^2 - \lambda I \mathbf{w} = 0$$

or

$$\begin{bmatrix} \gamma g_s - k^2 & -\gamma g_a \\ \gamma s_g & \gamma s_g - d k^2 \end{bmatrix} - \lambda = 0$$

so conditions are either

$$(\gamma g_s - k^2) + (\gamma g_a - d k^2) > 0$$

$$\delta (g_s + s_a) - (1 + d) k^2 > 0 \Rightarrow \mathbf{w} \text{ always negative}$$

or

$$(\gamma g_s - k^2)(\gamma s_g - d k^2) - \gamma^2 s g a < 0$$
or

\[ dh^4 - \delta (d g_s + 5a)^2 x^2 (g_s f a - \delta \tilde{g} a) < 0 \]

so \( dh > 0 \), last term > 0,
so at minimum,

\[ d g_s + 5a > 0 \], but \( g_s + 5a < 0 \),
so \( d \neq 1 \), \( g_s, 5a \) diff. signs.

Final condition:

\[ d g_s + 5a > 2 \sqrt{d (g_s f a - \delta \tilde{g} a)} > 0 \]

so all conditions for diffuse instability:

\[
\begin{align*}
& g_s + 5a < 0 \\
g_s f a - \delta \tilde{g} a > 0 \\
d \neq 1, \text{ } 5a, g_s \text{ diff. signs.} \\
d g_s + 5a > 2 \sqrt{d (g_s f a - \delta \tilde{g} a)} > 0
\end{align*}
\]
In 1 dimension,

\[ \nabla^2 W_k = \frac{\partial^2 W_k}{\partial x^2}, \quad \text{so} \quad W_k = A \sin kx + B \cos kx \]

and boundary conditions imply

\[ A = 0, \quad k = \frac{n \pi}{L} \]

In higher dimensions, tougher

thus, we substitute in \( w \), and get

\[ \lambda w = \gamma J w - Dk^2 w \]

so \( w \) is a solution if

\[ \left| \lambda w - \gamma J + Dk^2 \right| = 0 \]

\[ \begin{vmatrix}
\lambda - \gamma \delta_5 + Dk^2 & -\gamma g\delta_a \\
-\gamma S_5 \delta_0 & \lambda - \gamma S_5 + Dk^2
\end{vmatrix} = 0 \]

But now we want \( \lambda > 0 \) for some \( k^2 \)

so that our disturbances can form patterns.
by multiplying out, we obtain

\[-\gamma g_a f_s + \lambda + \gamma g_s f_a + dh^4 - \gamma \lambda (f_a + g_s) + \lambda (h^2 + dh^2)\]

\[-\gamma g_s dh^2 - \gamma f_a h^2\]

\[\lambda + \lambda \left[ k^2 (1+d) - \gamma (f_a + g_s) \right] + h(k^2) = 0\]

\[h(k^2) = dh^4 - \gamma (dg_s + g_s) h + \gamma^2 (g_s f_a - g_a f_s)\]

so if \( \lambda \) is to be negative, either

\[\left[ k^2 (1+d) - \gamma (f_a + g_s) \right] > 0\] * different from book

or

\[h(k^2) < 0\]

but \( (g_s f_a - g_a f_s) \) is \( > 0 \), so

we must have

\[dg_s + f_a > 0\]. But \( g_s + f_a < 0 \), so

we must have

\[d \neq 1\] and \( g_s, f_a \) have opposite signs.
so first we want to make sure the system is stable without diffusion

so we want

\[ s_t = \gamma g(s,a) \] to be stable
\[ a_t = \gamma f(s,a) \]

what do we do? [linear analysis.]

\[ J = \begin{pmatrix} \gamma g_s & \gamma g_a \\ \gamma f_s & \gamma f_a \end{pmatrix} \] solutions \( e^{\lambda t} \) where \( \lambda \) is an eigenvalue

\[ \text{i.e.} \]

\[ (\gamma g_s - \lambda)(\gamma f_s - \lambda) - \gamma g_a \gamma f_a = 0 \]

or \[ \lambda^2 - (\gamma g_s + \gamma f_s) \lambda + \gamma g_a \gamma f_a = 0 \]

or \[ \lambda^2 - (g_s + f_a) \lambda - g_s g_a = 0 \]

\( \lambda = 1, \quad B = -(g_s + f_a), \quad C = -g_s g_a + g_s f_a \)

so \[ \lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2} \]
Now, we see that we must have $A < 0$ for stability, so

\[ g_s + S_a < 0 \]

and also

If $-4AC$ is positive $\sqrt{\frac{11}{B_1}} > B$, so $A$ positive.

Thus, $-4AC =$ negative

so $C =$ positive, $g_s f_a - S_g a > 0$

thus

\[ \text{conditions for stability w/o diffusion:} \]

\[ g_s + S_a < 0 \]

\[ g_s f_a - S_g a > 0 \]

Now we look at what happens with diffusion:

still around steady state:

guess \( w_+ = e^{\gamma t} W_k(\vec{r}) \), where $W_k$ satisfies:

\[ \nabla^2 W_k + k^2 W_k = 0 \]
14.6

If the population is to die out,

\[ u^* = 0 \] must be an attractor

\[ u^* = \frac{(r-E)}{r} \] must be unstable

we then linearize the ODE around \( u^* \):

\[ u' = f'(u^*)u' + Du'_{xx}, \quad u = u' + u^* \]

\[ = \left( (r-E) - \frac{2ru^*}{K} \right) u' + Du'_{xx} \]

we guess a solution of the form

\[ u' = e^{\lambda t} (A \sin kx + B \cos kx) \]

so the boundary conditions yield:

\[ u'_x(0,t) = e^{\lambda t} K (A \cos 0 - B \sin 0) = e^{\lambda t} kA = 0 \]

or \( A = 0 \)
14.6 Consider the following system

\[ u_t = ru(1 - \frac{u}{K}) - EU + Du_{xx} \]

with

\[ u_x(0,t) = 0, \quad u(H,t) = 0 \]

and \( r > 0, \quad r - E > 0, \quad D > 0 \)

The homogeneous steady state is given by

\[ 0 = ru^*(1 - \frac{u^*}{K}) - Eu^* \]

\[ 0 = (r - E)u^* - \frac{ru^*}{K} \]

\[ 0 = ((r - E) - \frac{ru^*}{K})u^* \]

so either

\[ u^* = 0 \] or \( u^* = \frac{(r - E)r}{K} \)
and
\[ u'(H,t) = e^{\lambda t} B \cos kH = 0 \quad \text{or} \quad kH = \frac{n\pi}{2}, \quad n \text{ odd} \]
or \[ k = \frac{n\pi}{H^2}. \]

Then upon substituting back, we obtain
\[ \lambda = \left( (r-E) - \frac{2ru^*}{k} - D \left( \frac{n\pi}{2H} \right)^2 \right) \]

so for \( u^* = 0 \)
\[ \lambda = (r-E) - \frac{D(n\pi)^2}{4H^2} \quad \text{and} \quad \lambda < 0 \quad \text{when} \]
\[ H < \left( \frac{D}{r-E} \right)^{\frac{1}{2}} \frac{\pi}{2} \]
so this is one necessary condition for the population to die out.

For \( u^* = \left( \frac{r-E}{r} \right) k^* \),
\[ \lambda = -(r-E) - \frac{D(n\pi)^2}{4H^2} \quad \lambda > 0 \quad \text{when} \]
Consider the spruce budworm model:

\[ u_t = ru(1 - \frac{u}{q}) - \frac{u^2}{1 + u^2} + \partial u_{xx}, \quad u = 0 \text{ on } x = 0, L \]

we let \( x \to \frac{x}{L} \), and look for \( u \) s.t. \( \frac{du}{dt} = 0 \)

so \( f(u) + \frac{D}{L^2} u_{xx} = 0 \) or \( L^2 f(u) + D u_{xx} = 0 \)

we seek an \( L \) such that \( \exists 3 \) distinct roots.

so we note that \( u \approx A \sin(\pi x) \)

and we obtain

\[ L^2 f(u) - D \pi^2 u \approx 0 \]

or \( \frac{D \pi^2 u}{L^2} \approx f(u) = ru(1 - \frac{u}{q}) - \frac{u^2}{1 + u^2} \)
\[ f'(u) = r - \frac{2ru}{q} - \frac{(1+u^2)2u - u^2(2u)}{(1+u^2)^2} \]

\[ = r - \frac{2ru}{q} - \frac{2u}{(1+u^2)^2} \]

So \[ f'(u) = \frac{D\pi^2}{L^2} \] and \[ \frac{D\pi^2 u}{L^2} = f(u) \]

at \( L_0 \) or when

\[ \frac{D^2\pi^2}{L^2} = r - \frac{2ru}{q} - \frac{2u}{(1+u^2)^2} \]

\[ \frac{D^2\pi^2}{L^2} = r(1 - \frac{u^2}{q}) - \frac{u}{(1+u^2)^2} \]

or \[ \frac{ru}{q} + \frac{2u}{(1+u^2)^2} - \frac{u}{(1+u^2)} = 0 \]

or \[ \frac{r(1+u^2)^2 - (1+u^2) + 2 = 0}{q} \Rightarrow (1+u^2)^2 - \frac{q}{r}(1+u^2) + \frac{2q}{r} = 0 \]

\[ u^4 + \left( \frac{2r}{q} - 1 \right)u^2 + 2 + \frac{r}{q} = 0 \]
we have the reaction diffusion system:

\[ S(x,t) = \text{concentration of substrate} \]

\[ A(x,t) = \text{concentration of cosubstrate} \]

In the active layer, \( S \) and \( A \) react at the rate

\[ R = \frac{V_m AS}{K_m + S + S^2/K_S} \]

and \( S \) and \( A \) are fed by a reservoir at concentration \( S_0 \) and \( A_0 \) and diffuse,

\[ S_t = \frac{D_S}{L_1 L_2} (S_0 - S) - R + D_S \nabla^2 S \]

\[ A_t = \frac{D_A}{L_1 L_2} (A_0 - A) - R + D_A \nabla^2 S \]

if we define

\[ s = \frac{S}{K_m}, \quad a = \frac{A}{K_m}, \quad \tau = \frac{D_S}{L^2}, \quad \nabla^2 = L^2 \nabla^2 \]
then we get \( S = K_m S, \) \( A = a_k m, \)

\[ t = \frac{L^2}{D_S} \Rightarrow \frac{dt}{D_S} = \frac{L^2}{D_S} dt^*, \quad \nabla^2 = \frac{\nabla^*}{L^2} \]

\[ S_* = \frac{D_S K_m}{L^2} \frac{ds}{dt^*}, \quad A_* = \frac{D_S K_m}{L^2} \frac{da}{dt^*} \]

\[ R = \frac{V_m K_m s a}{K_m + K_m s + K_m s^2/K_5} = \frac{V_m K_m s a}{1 + s + K_m s^2} \]

\[ S_+ = \frac{L^2}{D_S K_m} \left[ \frac{D_S K_m (s_0 - s)}{L^2 L_2} - \frac{V_m K_m s a}{1 + s + K_m s^2} + \frac{K_m D_S \nabla^2}{L^2 K_5} \right] \]

\[ a_+ = \frac{L^2}{D_S K_m} \left[ \frac{D_a K_m (a_0 - a)}{L^2 L_2} - \frac{V_m K_m s a}{1 + s + K_m s^2} + \frac{K_m D_a \nabla^2}{L^2} \right] \]
\[ S_+ = \frac{L^2}{L_1 L_2} \left( s_0 - s \right) - \frac{L^2 V_m}{D_s} F(s, a) + \nabla s \]

\[ a_+ = \frac{L^2}{L_1 L_2} \left( a_0 - a \right) - \frac{L^2 V_m}{D_s} F(s, a) + \frac{D_a}{D_s} \nabla^2 a \]

where \( F(s, a) = \frac{s a}{1 + s + K s^2} \), \( K = \frac{K_m}{K_s} \).

or

\[ S_+ = \frac{L^2}{L_1 L_2} \left[ s_0 - s - \frac{L^2 V_m}{D_s} F(s, a) \right] + \nabla^2 s \]

\[ a_+ = \frac{L^2 D_3}{L_1 L_2 D_s} \left[ \frac{D_a}{D_s} (a_0 - a) - \frac{L^2 V_m}{D_s} F(s, a) \right] + \frac{D_a}{D_s} \nabla^2 a \]

and if \( y = \frac{L^2 D_3}{L_1 L_2 D_s}, \quad \alpha = \frac{D_a}{D_s}, \quad \beta = \frac{D_a}{D_s} \)

\[ \rho = \frac{L_1 L_2 V_m}{D_s} \]
11.2.1
then we obtain
\[ s_+ = \gamma g(s, a) + \nabla^2 s, \quad a_+ = \gamma f(s, a) + \beta \nabla^2 a \]
where
\[ g(s, a) = (s_0 - s) - \rho F(s, a) \]
\[ f(s, a) = \alpha(a_0 - a) - \rho F(s, a) \]
\[ F(s, a) = \frac{Sa}{1 + S + KSs^2} \]
and
\[ K = \frac{K_m}{K_s}, \quad \alpha = \frac{D_0}{D_s}, \quad \beta = \frac{D_0}{D_s}, \quad \rho = \frac{L_1L_2V_m}{D_s} \]
\[ \gamma = \frac{L^2 D_s}{L_1L_2 D_0} = \frac{D_s}{L_1L_2} \left/ \frac{D_s}{L^2} \right. \]
11.21

A homogeneous steady state \((\tilde{s}, \tilde{a})\) has

\[ g(\tilde{s}, \tilde{a}) = \tilde{s}(\tilde{s}, \tilde{a}) = 0 \]

i) \[ \dot{O} = (\tilde{s}_0 - \tilde{s}) - \rho F(s, a) \]

ii) \[ \dot{O} = \alpha(a_0 - a) - \rho F(s, a) \]

\[
\frac{\rho s a_0}{1 + s + ks^2} = \tilde{s}_0 - \tilde{s} \Rightarrow a = \frac{s_0 - s}{s} \left(1 + s + ks^2\right)
\]

(from i.)

so from ii.)

\[ \dot{O} = \alpha a_0 - \alpha a - \rho s a_0 \frac{\rho s a_0}{1 + s + ks^2} \]

\[
\dot{O} = \alpha a_0 - \alpha \left(\tilde{s}_0 - \tilde{s}\right) \left(1 + s + ks^2\right) - \tilde{s}_0 - \tilde{s}
\]

\[ \dot{O} = \rho s a_0 - \alpha \left(\tilde{s}_0 - \tilde{s}\right) \left(1 + s + ks^2\right) - \rho s \left(\tilde{s}_0 - \tilde{s}\right) \]
11.21

The Jacobian is given by

\[ \mathbf{J} = \begin{pmatrix} g_s & g_a \\ s_s & s_a \end{pmatrix} \]

where

\[ g_s = -1 - \rho F_s = -1 + \rho \left[ \frac{(1+s+k s^2)a - sa(1+2ks)}{[1+s+k s^2]^2} \right] \]

\[ g_a = 0 - \frac{\rho s}{1+s+k s^2} < 0 \]

\[ s_s = 0 - \rho \frac{a-k s^2}{(1+s+k s^2)^2} \]

\[ s_a = -\alpha - \frac{\rho s}{1+s+k s^2} < 0 \]
Clearly, \( g_s, f_a < 0 \) because \( F_a, \alpha, \epsilon > 0 \) and \( 1 > f_a, 1 > g_s \).

The conditions for instability are then:

\[
g_s + f_a < 0, \quad g_s f_a - g_a f_s > 0
\]

At the steady state:

Since \( g_a, f_a < 0, 1 > g_a, 1 > f_a \) and we have

\[
-g_s f_a + g_a f_s > 0 \quad \text{or} \quad g_a f_s > g_s f_a \quad \text{or} \quad \frac{g_a}{f_a} > \frac{g_s}{f_s}
\]

If \( f_s > 0 \), we have:

\[
\frac{g_s}{f_s} > \frac{g_a}{f_a} \Rightarrow \frac{e^F_a}{\alpha + \rho F_a} > -1 - \rho F_s = 1 + \frac{1}{\rho F_s}
\]

But \( F_a > 0 \), so LHS is always less than 1 so we have:

\[
1 > 1 + \frac{1}{\rho F_s} \quad \text{or} \quad 0 > \frac{1}{\rho F_s} \quad \text{or} \quad F_s < 0
\]