Flat-Foldability of Origami Crease Patterns

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1 FLAT ORIGAMI.

Origami has traditionally been appreciated as an art form and a recreation. Increasingly over the course of the 20th century, however, attention has been drawn to the scientific and mathematical properties of paperfolding, with the majority of this work occurring in the past 20 years or so. Even today, many of the most basic and intuitive problems raised by origami still lack definitive solutions.

Origami is unlike most forms of sculpture in that its medium—a sheet of paper, usually square in shape—undergoes almost no physical change during the creation process. The paper is never cut nor chemically manipulated; its size, shape, and flatness are never altered; nothing is ever added or taken away. Only its position in space is affected. Origami has been described as an "art of constraints." The art lies in exploring and expanding the realm of what can be achieved within the constraints naturally imposed by the paper. Designing an origami model of a particular subject requires considerable ingenuity.

Many origami models are so cleverly designed that their final forms bear almost no resemblance to the sheets of paper from which they are made. However, there exists a substantial class of origami forms which share one basic characteristic of the original paper: flatness. Flat models like the crane **(fig. 1)** can be pressed between the pages of a book. A mathematical inquiry into paperfolding could logically start by examining the rules that govern flat-folding. The added constraint of flatness actually simplifies our mathematical description of folding by reducing the number of relevant spatial dimensions to two. In reality, of course, flat models are truly 3-D. There is a narrow space separating overlapping layers, making it possible to distinguish the order in which the layers are stacked. As we shall see later on, it is nonetheless possible to describe a flat origami as a 2-D mathematical abstraction without losing any information about the overlap order of its layers.

2 CREASE PATTERNS.

Before we proceed to formulate a mathematical description of flat-folding, let us pause to consider an example. When we open out a paper crane and look at the square from which it was folded, we see a web of crease lines crisscrossing all over the paper **(fig. 2)**. This web is the model's *crease pattern*. It bears little resemblance to a crane, and it seems somehow surprising and mysterious that the two forms are connected somehow. Developing a mathematical theory that relates the crease pattern to the folded form shall be our main purpose in this study.

The marks on the paper fall into three distinct types. Some of the crease lines are left over from developmental stages of the folding process and do not correspond to any folded edges in the finished model; these lines we ignore, because we are only interested in creases used in the folded form. The remaining lines are either *mountain creases* or *valley creases*. A mountain crease is left by a fold that moved paper away from the folder; when I set an open book face down on a table, the spine is a mountain crease. A valley crease is left by a fold that moved paper towards the folder; when I am reading a book, the binding is a valley crease.

Let us translate these observations into mathematical abstractions by giving a few definitions.

Definition 1. The *sheet*, S, shall be defined as a compact connected region of the plane, bound by a smooth simple closed curve.

Note that a sheet must have finite area by this definition. Since most art supply stores do not stock infinite-area origami paper, this convention seems realistic. However, the mathematical theory of paperfolding can be logically extended to the infinite case. For a discussion of infinite-area origami, see Justin.

Definition 1 does not allow the paper to have holes. In this study we shall assume, unless stated otherwise, that every sheet has exactly one boundary, so it is simply connected. This seemingly arbitrary assumption has the effect of further restricting what is possible to fold, thus simplifying the math required for describing flat origami. Fortunately, most real origami paper we use has no holes, so our assumption is a good approximation of reality. For a treatment of the general case in which paper may have multiple boundaries, see Justin.

Definition 2. The crease pattern, G, shall be defined as a planar graph embedded on S. The area of S is entirely partitioned by the vertices, edges, and faces of G. The entire boundary of S is partitioned into edges and vertices. An edge lying in the interior of S is called a *crease*, whereas an edge on the boundary of S is called a *raw edge*.

For our study let us always assume that G is a finite graph. Infinite crease patterns are theoretically possible, but again this possibility complicates the math and is a poor approximation of the reality of origami, so we ignore it here. To learn about folding infinite crease patterns, please refer to Maekawa.

Definition 3. Given a vertex V in a crease pattern, we denote the set of edges having

V as an endpoint by $E(V)$, and the set of faces having V on their boundaries by $F(V)$. In general, when working with these sets we shall name the elements in clockwise sequential order **(fig. 3)**. (The designation F_0 is assigned arbitrarily if V is an interior vertex.)

Definition 4. A *c-net*, C, is a sheet of paper S with a crease pattern G embedded on it. The elements of C and S are the same—they are the points of the sheet S.

Definition 5. Recall that each crease corresponds to one of two types of crease lines, mountain or valley. Define an *MV-assignment* to be a map from the set of creases in a crease pattern to the set $\{M, V\}$. A crease pattern together with an MV-assignment shall be called a *signed* crease pattern; a crease pattern without an MV-assignment is called *unsigned*. Similarly, c-nets may also be signed or unsigned.

3 FLAT-FOLDABILITY.

Consider the signed crease pattern of the crane. Somehow, this arrangement of mountain and valley folds work together to collapse the paper into a flat origami which happens to resemble an avian life form. By contrast, consider the unsigned crease pattern in **figs. 4 & 5**. No matter how we try, these patterns cannot be folded flat—some layer of paper always ends up blocking another. It doesn't even matter what MV-assignment we use, the paper still stubbornly refuses to fold flat. Apparently, the unsigned c-net of the crane has some fundamental property that ensures the existence of an MV-assignment which enables the paper to fold flat. We call this property *flat-foldability* of an unsigned c-net.

Suppose we are given a flat-foldable unsigned c-net such as that of the crane or some other flat model. It it possible to predict what the final model will look like? To what extent does the crease pattern determine where each part of the sheet goes? In our study of paperfolding, we will formulate a mathematical model of paperfolding that enables us to answer the above questions. The main goal of this study is to answer the following question.

Main Problem: Is a given unsigned c-net flat-foldable?

We want to give the "flat foldability" property a mathematical definition, stated in terms of the known geometric properties of an unsigned c-net. This definition should model our real experience of paperfolding as nearly as possible.

A flat origami model is really three-dimensional, so it makes sense initially to treat flat origami as a special case of 3-D origami. In the next section, we develop a set of rules for 3-D paperfolding, which will function as a starting point for developing a mathematical simulation of 2-D paperfolding.

4 FOUR AXIOMS FOR POLYHEDRAL FOLDING.

Our formal definition of 3-D folding must necessarily spring from empirical observations of the folded paper. To begin, observe that folding moves every point of the sheet through space to a final location in the folded model. It makes sense to imagine folding as a mapping ϕ from the c-net C into 3-space. The image $\phi(C)$ represents the folded model.

Paperfolding is the art of moving paper without violating the constraints naturally imposed by the material. To complete the definition of ϕ, a list of these constraints is required. The following list of four constraints is complete. Each one of them is easily formalized as a mathematical axiom.

Notation. In general, if f is a function and Y is any subset of the domain of f, we shall denote the restricted function by $f|Y$.

ONE. The crease pattern faces retain their shapes when the paper is folded. Paper is a stiff and inelastic material, so each face remains flat and its area always stays the same. Folding may translate or rotate a face through space, but folding never distorts the face's shape or changes its size. This suggests our first axiom.

Axiom 1. ϕ acts as an isometry on each face.

Definition 6. The *face isometry* $i_F : \mathbb{R}^3 \to \mathbb{R}^3$ is the isometry as which ϕ acts on the face F. That is, $\phi|F = i_F|F$.

Note that axiom 1 assumes that the paper is not allowed to curl. To account for the models in which the paper is curled, axiom 1 would have to be replaced with a weaker restriction against altering the intrinsic geometry of each face. Models in which the paper curls are called *developable surfaces*. Models in which the faces remain flat are called *polyhedral origamis*. Flat origami is polyhedral, since all the paper winds up lying flat over a plane. The main focus of this study is flat origami, so we shall assume that folding acts as an isometry on each face.

TWO. The paper is folded at every crease. This constraint is not imposed by the material; it simply describes what happens to the crease pattern when we fold it. The first constraint disallows folding the paper where there is no crease; the second constraint requires folding the paper where there is a crease. Now, when we make a fold, the paper on one side of the crease moves in relation to the paper on the other side. In light of definition 6, we can express this phenomenon as follows:

Axiom 2. If F_1 is adjacent to F_2, then $i_{F_1}|C \neq i_{F_2}|C$.

THREE. In the folded model, the paper cannot self-intersect. After all, paper is solid tangible matter. Even if the origami is “flat”, there is still a small space between overlapping layers. If the folded paper were allowed to self-intersect, then a point of intersection would be the image under ϕ of multiple points in the c-net. So, what this constraint is really saying is:

Axiom 3. ϕ is one-to-one.

Technically, axiom 3 prohibits any fold from being truly flat. The dihedral angle of any fold cannot go to zero without violating the one-to-one axiom. When we discuss flat-folding, the 3-D folding map ϕ will no longer work. We shall use a slightly modified version of ϕ, called ω, for flat-folding. For the moment, however, we shall continue to discuss 3-D origami that satisfies all these constraints.

FOUR. The paper never tears. Many old-fashioned origami models involve cutting, but most modern folders believe that cutting disrupts the purity of the square sheet. Thus we shall assume that folding leaves the shape of the paper unaltered. Note that constraint #1 automatically prohibits slicing apart the area of a face. Constraint #4 prohibits cutting along creases as well. Folding cannot cause adjacent faces to become separated; if this happened, the folding map would be discontinuous at all points on the crease where the cut was made. In general, any discontinuity in folding corresponds with a cut or tear made in the paper. Thus our last axiom is:

Axiom 4. ϕ is continuous.

Axioms 1 through 4 form a complete description of the constraints on polyhedral origami, assuming the paper is simply connected as required by definition 1. However, if we had defined the sheet of paper more generally so that it could have multiple boundary curves, we would require one additional axiom to ensure that the image of those closed curves under ϕ is unlinked. **Figure 6** shows an impossible fold on a paper with two boundaries; the boundaries are linked in the folded image, so the fold violates the fifth axiom. It can be shown that this fifth requirement is independent of the first four axioms. See Justin for a treatment of origami paper with holes. In this study we shall assume that the paper has no holes and make do with only four axioms.

It is satisfying to note that axioms 3 and 4, together with the compactness of C, imply that ϕ is a homeomorphism between the sheet of paper C and the folded origami model $\phi(C)$. This is as it should be, for in real life origami paper rarely undergoes topological changes during the folding process! A proof of this result is easy but outside the scope of this report.

5 BREAKING THE PROBLEM DOWN.

The above four constraints seem to comprise a complete description of the physical limitations of polyhedral paperfolding. Any mapping of a c-net into 3-space satisfying all four axioms is a realistic mathematical simulation of folding. However, these axioms do not allow for truly flat origami. If we tried to fold a crane according to these axioms, the folds would all have small but nonzero dihedral angles, causing the model to puff up into the third dimension. Although this mathematical model is more physically accurate, it is not useful for determing the flat-foldability of crease patterns. Therefore, our mathematical simulation of flat-folding will use a different strategy.

Since the folded model will lie flat, the final position of each point on the paper can be completely described by two properties: first by where it lies over the 2-D plane; second by the number of layers that lie directly underneath it. Therefore, we could define flat-folding as a map ω from the c-net C to the space $\mathbb{R}^2 \times \mathbb{Z}$. The real number coordinates are the x and y position of the point, and the positive integer refers to how many layers of paper lie underneath this point.

In choosing to use ω instead of ϕ for flat-folding, we give up a certain amount of realism. Whereas ϕ never causes the paper to self-intersect, ω forces entire regions of the paper to occupy the same space in the plane when they are folded over one another. However, ω is still one-to-one, because two points in the c-net that get mapped to the same point on the plane cannot be in the same layer of paper. Unfortunately, ω is not continuous; the integers are discrete, so if the folded form has more than one layer of paper, each layer is separated from the others in the topology of $\mathbb{R}^2 \times \mathbb{Z}$, so the image is not connected. Only in the trivial case where C has no creases is ω continuous.

We can easily get around this problem by breaking ω down into its component parts. We define a *semifolding map* $\mu : C \to \mathbb{R}^2$ that determines only the final position of the paper in the plane, and a *superposition ordering* $\sigma : C \to \mathbb{Z}$ that determines only the overlap order of the layers. We then define the flat-folding map ω as the cartesian product of the semifolding map and the superposition ordering.

6 FOLDING WITH SELF-INTERSECTION: SEMI-FOLDING.

Before we can study the order in which the layers overlap, we must first determine which parts of the paper wind up overlapping. Thus we must precisely specify μ before σ. **Fig. 7** illustrates the effect of μ on a c-net. Suppose a paper crane was folded from paper that can pass through itself. The final folded form can collapse into two dimensions by allowing each stack of overlapping layers to occupy one common region in the plane. The resulting bird-shaped silhouette is the semifolding image of the crane's c-net. Thus μ is an immersion of C into the plane.

Semifolding is not one-to-one—the paper self-intersects wherever there are overlaps—so in defining μ we are free from the constraint of axiom 3. We insist that μ must abide by axioms 1 and 2, so that faces cannot deform and every crease gets a fold. However, we shall not guarantee that the semifolding map is continuous on all crease patterns. If it is possible for a c-net C to collapse flat without cutting when folded with self-penetrable paper, then μ should be continuous on C; but if this is not possible, μ is defined on C discontinuously. In other words, when semifolding we can cut the paper if we absolutely have to.

Definition 7. A c-net C is said to be *semifoldable* if there exists a mapping $\mu : C \to \mathbb{R}^2$ which satisfies axioms 1, 2, and 4. (Unfortunately this terminology could stand some improvement. Note that a semifolding map is defined on C whether or not C is semifoldable—the difference is in the continuity.)

In general, we require μ to abide by axiom 1, so it acts on each face F by a face isometry i_F. However, the range of μ is restricted to the plane, so i_F is a planar isometry. We assume henceforward that face isometries are isometries of $\mathbb{R}^2$.

Let us now derive results concerning the semifoldability of crease patterns. Our first result states that the creases in a foldable crease pattern are straight line segments. The theorem assumes only axioms 1, 2, and 4, so the theorem and proof may be generalized to polyhedral folding without self-intersection. However, for simplicity, the statement and proof presented here applies specifically to semifolding, so the map is assumed to be 2-D.

First we need a brief lemma.

Lemma 1. Let $f : C \to \mathbb{R}^2$ be a function satisfying axiom 1. Let E be the crease between adjacent faces F_1 and F_2 in C. Then f is continuous on E if and only if,

$$f|E = i_{F_1}|E = i_{F_2}|E.$$

Proof. Suppose f is continuous on E. Let x be a point on E, and let $\{x_1, x_2, ...\}$ be a sequence of points in F_1 that converges in C to x. Since the isometry i_{F_1} is continuous, the image of this sequence under f converges to the point $i_{F_1}(x)$. Since f is continuous at x, we have $f(x) = i_{F_1}(x)$ for all $x \in E$. Thus, $f|E = i_{F_1}|E$. The same argument can be applied for F_2.

For the converse, suppose $f(x) = i_{F_1}(x) = i_{F_2}(x)$ for all points x on E. Let Y be an open neighborhood in $f(C)$ of the point $f(x)$. Let $U = f^{-1}(Y)$ be the preimage of Y. By axiom 1, we have

$$U = (i_{F_1}^{-1}(Y) \cap F_1) \cup (i_{F_2}^{-1}(Y) \cap F_2) \cup (f^{-1}(Y) \cap E)$$

Our hypothesis states that f agrees with the face isometries i_{F_1} and i_{F_2} on the edge E. This implies that the three sets whose union is U fit together nicely on C. Since the face isometries themselves are continuous, the preimage of Y is an open neighborhood of x. Therefore, f is continuous. $\square$

Theorem 2. Let E be the crease between adjacent faces F_1 and F_2 in a c-net C. If there exists a semifolding map $\mu : C \to \mathbb{R}^2$ which is continuous on E, then E is a straight line segment.

Proof. By the lemma, we have $i_{F_1}|E = i_{F_2}|E$. Compose both sides of this equation with the isometry $i_{F_1}^{-1}$ to get

$$I|E = i_{F_1}^{-1} \bullet i_{F_2}|E$$

where $I|E$ is the identity function on the edge E. That is, the composition $i_{F_1}^{-1} \bullet i_{F_2}$ holds all points of edge E fixed. Clearly the composition $i_{F_1}^{-1} \bullet i_{F_2}$ is itself an isometry of $\mathbb{R}^2$ Now, there are two types of planar isometries that hold more than one point fixed: one is the identity map; the other is a reflection in a line containing all the fixed points. By axiom 2, $i_{F_1} \neq i_{F_2}$, so $i_{F_1}^{-1} \bullet i_{F_2}$ is not the identity on $\mathbb{R}^2$. Therefore, $i_{F_1}^{-1} \bullet i_{F_2}$ is a reflection in a line containing the edge E. Hence, E is a straight line segment. $\square$

As previously mentioned, theorem 2 can be generalized to the polyhedral folding map ϕ. The proof of the 3-D case must take into account several different spatial isometries that hold multiple points fixed; in each of these cases the fixed points turn out to be colinear, so the result holds. The details are left as an exercise to the reader.

The following corollary to theorem 2 is important enough to be called a theorem in its own right. It states that semifolding flips over one face in every adjacent pair.

Theorem 3. Let F_1 and F_2 be adjacent faces in a c-net C, with the crease E separating them. If μ is continuous at E, then μ acts as a reflection or a glide reflection on exactly one of F_1 and F_2.

Proof. The proof of the preceding theorem showed that $i_{F_1}^{-1} \bullet i_{F_2}$ is a reflection. Since their composition is orientation-reversing on the plane, exactly one of the face isometries i_{F_1} and i_{F_2} is orientation-reversing. The result follows immediately. □

One result of theorem 3 is the following necessity condition for semifoldability.

Theorem 4. If μ is continuous in the neighborhood of a given interior c-vertex V, then V is of even degree. Therefore, if a c-net is semifoldable, then each of its interior c-vertices is of even degree.

Proof. Let d be the degree of V. When $d = 0$, there are no creases at V, so V is really a point in the interior of a c-face, where axiom 1 guarantees that μ is continuous. Thus we may assume that $d > 0$. Furthermore, d cannot be 1 because then the single crease radiating from V bounds the same face on both sides; this face's self-adjacency violates axiom 2. Thus we may assume that $d > 1$. To each face in $F(V)$ which μ does not reflect, assign the label $+1$; to the remainder, assign the label -1. By theorem 3, adjacent faces cannot have the same label. Consider all pairs of adjacent faces $F_i, F_{i+1 \bmod d}$ which are labeled $+1$ and -1, respectively. Clearly, every face at V is a member of exactly one such pair; thus, these pairs partition $F(V)$. It follows that d equals 2 times the number of pairs, so d is even. □

The result does not hold if the crease pattern is not semifoldable. Interestingly, vertices of any degree are possible in non-flat origami; see Hull/Belcastro for information.

An important corollary to theorem 4 concerns face colorings of c-nets. A planar graph is called *face k-colorable* if it is possible to color each face with one of k different colors, such that adjacent faces are never given the same color. A face 2-colorable graph, for example, can be colored black and white like a chessboard.

Corollary 5. If a c-net is semifoldable, then its crease pattern is face 2-colorable.

Proof. (Adapted from Hull, T.) Let C be a semifoldable c-net. By the above theorem, each interior c-vertex in C has even degree. Therefore, C is eulerian, hence 2-face colorable. □

Consider the significance of this corollary. The proof of the theorem makes clear that the color assigned to a face is dependent on whether or not μ reflects the face. The act of semifolding flips over all the c-faces with one color, and merely rotates or translates the others. This is consistent with our experience of paperfolding. In the paper crane, for instance, we find that every other face in the crease pattern lies face up in the folded model.

7 DEFINING μ.

We still have not formulated a precise definition of μ. We may assume μ holds the position of at least one face constant, so let us arbitrarily pick one c-face in C, call it F_{fix}, and require that $\mu|F_{\text{fix}} = I$, the identity on F_{fix}. The rest of the paper folds over, under, and around F_{fix}. The choice of which face to hold constant is made without loss of generality because for any c-face F, the entire image $\mu(C)$ may be moved via an isometry i such that $i \bullet \mu(F) = F$. Now consider a c-face F which is separated from F_{fix} by several creases and faces. Since the paper separating F from F_{fix} gets folded over and over along the creases existing in that region, it seems intuitively reasonable to assume that the final position of F is the image of F reflected through all of those creases.

Figs. 8 & 9 illustrate our motivation for making this assumption. Consider a rectangle with two non-intersecting creases, E_1 and E_2, separating the paper into faces F_1, F_2, and F_3. If we fold the rectangle along these creases while holding F_1 fixed, F_3 is reflected first over E_2, and then over E_1.

As a more complex example, suppose the crane is folded up around the triangle F_{fix} as shown. Then final location of the triangle F is accurately predicted by our assumption. Reflecting F over edges E_1, E_2, E_3, E_4 and E_5, in that order, places F directly over F_{fix}, just as it is positioned in the model. Note that other sequences of edge-reflections are possible. In the next section we will show that all of these sequences compose to give the same isometry if and only if the c-net is semifoldable.

The following two definitions formalize our assumption and precisely describe μ.

Definition 8. Let C be a c-net with straight-line creases. Let p be a vertex-avoiding path in C. The *isometry induced by* p, denoted $i_p : \mathbb{R}^2 \to \mathbb{R}^2$, is defined as follows. Suppose p crosses the creases $E_1, E_2, \ldots, E_n$, not necessarily all distinct, in that order. Let R_{E_j} denote the reflection of the plane across the line containing the crease E_j. Then i_p is defined to be the composition $R_{E_1} \bullet R_{E_2} \bullet \ldots \bullet R_{E_n}$.

Definition 9. Let C be a c-net with straight-line creases. A *semifolding map* $\mu : C \to \mathbb{R}^2$ is defined as follows: Choose any c-face F_{fix}, which shall be held constant by μ. For each c-face F, choose a vertex-avoiding path p in C from a point in the interior of F to a point in the interior of F_{fix}; we call p the *semifolding path* of F for μ. The resulting semifolding map is defined on each face F by $\mu(F) = i_p(F)$. If X is a point on an edge or vertex, and the limit of μ exists at X, then $\mu(X)$ is placed at that limit point.

Note that this definition technically depends on the choices of F_{fix} and the semifolding paths. As already explained, the choice of F_{fix} is ultimately inconsequential. The choice of semifolding paths, on the other hand, sometimes makes a significant difference, as we shall see.

8 THE ISOMETRIES CONDITION.

We now present the famous Kawasaki Theorem, also known as the isometies condition. Throughout the formula and the proof we use the following notation: Given a vertex V, A_i shall denote the angle of the corner of $F_i \in F(V)$ at V.

Theorem 6 (Kawasaki Theorem). Let C be a c-net with straight creases and even-degree interior vertices. The following five statements are equivalent:

(1) C is semifoldable; that is, there exists a semifolding map μ which is continuous on C.

(2) The alternating sum of the angles surrounding any interior c-vertex is 0; that is

$$A_0 - A_1 + A_2 - A_3 + \ldots - A_{d-1} = 0.$$

Note that the last term in the alternating sum is always negative because d is even.

(3) The sum of every other angle about an interior vertex V is 180°; that is

$$A_0 + A_2 + \ldots + A_{d-2} = A_1 + A_3 + \ldots + A_{d-1} = 180°.$$

(4) Let q be any closed vertex-avoiding path that starts and ends at a point in the interior of any c-face F. Then $i_q = I$, the identity.

(5) The definition of μ does not depend on the choice for each semifolding path; that is, for each c-face F, the isometry i_p is the same no matter what semifolding path p is used.

Remark. It seems intuitively reasonable that if an origami folds flat, the final position in the plane of a given face F is completely determined by the crease pattern and the choice of the fixed face F_{fix}. Parts (1) and (5) of the isometries condition confirms this suspicion: If μ is continuous on C for one choice of semifolding paths, it is continuous for every choice. Technically, definition 9 does not depend on any particular choice of semifolding paths; similarly, in statement (1) of the isometries condition there is only one unique possible semifolding map μ, up to isometry.

Remark. Note that, if we allow the sheet to have holes, statements (2) and (3) need to be modified slightly to account for would-be vertices lying in the sheet's holes. See Justin.

Proof. We will show (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1).

(1) ⇒ (2). Suppose C is semifoldable. If C has no interior vertices then (2) is trivially true, so we may assume that C has at least one interior vertex, V. Let μ be a semifolding map that acts as a non-reflection isometry on the face $F_0 \in F(V)$. Repeated application of theorem 3 then shows that μ reflects only the odd-numbered faces in $F(V)$.

We use the following notation: A_{E_i,E_j} shall refer to the angle between E_i and $E_j \in E(V)$; similarly, $A_{\mu(E_i),\mu(E_j)}$ is the angle between $\mu(E_i)$ and $\mu(E_j)$. We claim:

$$A_{\mu(E_0),\mu(E_k)} = \sum_{i=0}^{k-1} (-1)^i A_i \quad \text{for } 1 \leq k \leq d-1.$$

The proof of this claim is by induction on k. For the base case, set $k = 1$. Then $A_{\mu(E_0),\mu(E_1)}$ is just the angle at the corner of $\mu(F_0)$. That angle was unchanged by μ which acted as an isometry on F_0, so $A_{\mu(E_0),\mu(E_1)} = A_0$. For the inductive step, suppose the claim holds for $k = j$. If j is even, then μ does not reflect F_j, so its angle $A_j = A_{\mu(E_j),\mu(E_{j+1})}$ sweeps a counter-clockwise-pointing arc. In this case, the face $\mu(F_j)$ contributes additively to the total angle $A_{\mu(E_0),\mu(E_{j+1})}$; that is, $A_{\mu(E_0),\mu(E_{j+1})} = A_{\mu(E_0),\mu(E_j)} + A_j$. If j is odd, then μ reflects F_j, so its angle A_j sweeps a clockwise-pointing arc. In this case, the face $\mu(F_j)$ contributes subtractively to the total angle, so $A_{\mu(E_0),\mu(E_{j+1})} = A_{\mu(E_0),\mu(E_j)} - A_j$. These two cases show that the claim holds for all $k \leq d - 1$.

To complete the proof, consider the angle $A_{\mu(E_0),\mu(E_{d-1})}$, which equals the alternating sum of all the angles around V except for A_{d-1}. This last angle bridges the gap between E_{d-1} and E_0. When A_{d-1} is contributed to the total angle, the result is $A_{\mu(E_0),\mu(E_0)} = 0$. Since the degree of V is even, $d - 1$ is odd, so μ reflects F_{d-1}. Therefore, the face $\mu(F_{d-1})$ contributes subtractively to the total angle $A_{\mu(E_0),\mu(E_0)}$. This gives us

$$\sum_{i=0}^{d-1}(-1)^i A_i = A_{\mu(E_0),\mu(E_{d-1})} - A_{d-1} = A_{\mu(E_0),\mu(E_0)} = 0.$$

(2) $\Rightarrow$ (3). Suppose that the equation in statement (2) holds. For every odd $i < d$, add $2A_i$ to both sides of the equation. The result is:

$$A_0 + A_1 + A_2 + A_3 + \ldots + A_{d-1} = 2A_1 + 2A_3 + \ldots + 2A_{d-1}.$$

The left side of this equation is $360°$. Dividing by two yields:

$$180° = A_1 + A_3 + \ldots + A_{d-1}.$$

Finally, take the equation in (2) and add every odd-numbered angle once. This gives us:

$$A_0 + A_2 + \ldots + A_{d-2} = A_1 + A_3 + \ldots + A_{d-1}.$$

(3) $\Rightarrow$ (4). Suppose that (3) holds; that is, at each interior vertex the sum of every other angle is $180°$.

Consider the closed path q in C. It is possible to create a simple closed path q' by replacing each self-crossing of q with two non-intersecting pieces in one of two ways and reversing the direction on some sections of the path accordingly; see **fig. 10** for an example. (To see why this is always possible, consider q as a graph with fourth-degree nodes at each crossing. Then q' is an non-self-intersecting eulerian cycle.) Since q and q' both cross the same creases the same number of times, we have $i_q = i_{q'}$. Thus it will suffice to prove that statement (4) holds for any simple closed path q.

Let q be a simple closed path in C. Then q crosses the crease E an odd number of times if and only if E has one of its vertices lying in the interior of q and the other lying outside of q.

Let $v(q)$ be the number of c-vertices that lie in the interior of q. If $v(q) = 0$, then q crosses every crease an even number of times. Now consider the expression $i_q = R_{E_1} \bullet R_{E_2} \bullet \ldots \bullet R_{E_n}$. Each reflection R_{E_i} appears in this expression an even number of times. The reflections then cancel one another out in pairs, giving $i_q = I$, the identity. Thus statement (4) holds when $v(q) = 0$.

Now consider the case $v(q) = 1$. Let V be the single c-vertex lying inside of q. Every crease in $E(V)$ is crossed by q an odd number of times, since it has precisely one endpoint inside q. Every crease not in $E(V)$ is crossed by q an even number of times. All pairs of equal reflections in the expression of i_q cancel out, leaving $R_{E_0} \bullet R_{E_1} \bullet \ldots \bullet R_{E_{d-1}}$, for $E_i \in E(V)$. Thus i_q is the product of reflections in the creases in $E(V)$. Now, the product of the reflections in any pair of intersecting lines is a rotation about the point of intersection, through twice the angle separating the two lines. Let O_{A_i} represent a rotation of the plane through angle A_i about the point V. Thus we write:

$$
\begin{aligned}
&i_p = R_{E_0} \bullet R_{E_1} \bullet \ldots \bullet R_{E_{d-1}} \\
&= (R_{E_0} \bullet R_{E_1}) \bullet \ldots \bullet (R_{E_{d-2}} \bullet R_{E_{d-1}}) \\
&= O_{2A_0} \bullet O_{2A_2} \bullet \ldots \bullet O_{2A_{d-2}} \\
&= O_{2(A_0 + A_2 + \ldots + A_{d-2})}.
\end{aligned}
$$

By statement (3), the sum in the last expression is $2(180°) = 360°$. A rotation through 360°is the identity, so $i_p = I$ whenever $v(q) = 1$.

To prove the general case, we require the following observation. Let p_1 be any path in C, let p_2 be any path starting at the terminal point of p_1, and let $p_1p_2 = p_2 \bullet p_1$ be their composition. Then by definition 8, we have $i_{p_1p_2} = i_{p_2} \bullet i_{p_1}$.

We use induction on $v(q)$ to prove the case $v(q) > 1$. Assume that the theorem is true for any path that encircles k c-vertices. Let q be a simple closed path beginning and ending at point Q, such that $v(q) = k + 1$. Without loss of generality, suppose q points counter-clockwise. Let p be another simple closed path beginning and ending at Q, pointing clockwise, lying entirely in the interior of q, and encircling exactly one c-vertex. Then we have:

$$i_q = i_{qpp^{-1}} = i_{p^{-1}} \bullet i_{qp}.$$

Now, since p^{-1} encircles just one c-vertex, $i_{p^{-1}} = I$, as was just shown. Since

$$v(qp) = v(q) - v(p) = k - 1,$$

we have $i_{qp} = I$ by the induction hypothesis. Thus $i_q = I$ when $v(q) = k + 1$, and by induction for any closed path q.

(4) $\Rightarrow$ (5). Suppose statement (4) holds; that is, the product of reflections around a closed curve is the identity. Let p and p' be two semifolding paths for the face F. Both of these paths begin somewhere in F and end somewhere in F_{fix}. Draw a path from the beginning of p' to the beginning of p, contained entirely inside F, and call it q. Draw a path from the end of p to the end of p', contained entirely inside F_{fix}, and call it q' **fig. 11**. Since q and q' do not cross any creases, we have $i_q = i_{q'} = I$. Now, the combined path $pq'p'^{-1}q$ is closed, so $i_{pq'p'^{-1}q} = I$.

Recall that for any composition of paths $p_1 p_2$, we have $i_{p_1 p_2} = i_{p_2} \bullet i_{p_1}$. Therefore we can write:

$$i_{pq'p'^{-1}q} = i_q \bullet i_{p'^{-1}} \bullet i_{q'} \bullet i_p = I \bullet i_{p'^{-1}} \bullet I \bullet i_p = i_{p'^{-1}} \bullet i_p.$$

Thus $i_{p'^{-1}} \bullet i_p = I$. Compose both sides with $i_{p'}$ to get:

$$i_{p'} \bullet i_{p'^{-1}} \bullet i_p = i_{p'}.$$

Now, the left side reduces:

$$i_{p'} \bullet i_{p'^{-1}} \bullet i_p = i_{p'p'^{-1}} \bullet i_p = i_p.$$

Thus, $i_p = i_{p'}$. Therefore, μ does not depend on the choice of semifolding paths.

(5) $\Rightarrow$ (1). Suppose we have a c-net C and a semifolding map $\mu : C \to \mathbb{R}^2$. Suppose further that, if μ' is any other semifolding map having the same fixed face F_{fix}, then $\mu = \mu'$. We want to show that μ is continuous. Axiom 1 guarantees that μ is continuous throughout the interiors of the faces. If μ turns out to be continuous on every crease, then the limit of μ exists at the vertices, so by definition 9 μ is continuous at the vertices. Therefore we need only examine the creases. Let E be the crease between adjacent faces F_1 and F_2. Let p_1 be the semifolding path of F_1 for μ. Let q be a path from a point in F_2 to the starting point of p_1 in F_1, and suppose that q crosses no creases but E, which it crosses once. So $i_q = R_E$.

Let μ' be a semifolding map with the same fixed face as μ, and for which the semifolding paths of F_1 and F_2 are $p'_1 = p_1$ and $p'_2 = qp_1$, respectively. Note that:

$$i_{p'_2}(E) = i_{qp_1}(E) = i_{p_1} \bullet i_q(E) = i_{p'_1} \bullet R_E(E) = i_{p'_1}(E).$$

The equation $i_{p'_1}(E) = i_{p'_2}(E)$ shows that $\mu'(F_1)$ and $\mu'(F_2)$ have as a common edge $\mu'(E)$, thus implying μ' is continuous at E. By hypothesis, $\mu = \mu'$, so it follows that μ is also continuous at E. The same argument can be repeated for every crease E, so μ is continuous on all of C. $\square$

9 SUPERPOSITION ORDERING.

The isometries condition is a perfect tool for determining a c-net's semifoldability. All we have to do is check that every interior vertex satisfies the 180° condition—that is, the sum of every other angle around the vertex is 180°.

Semifoldability is not sufficient for flat-foldability, however. The c-net of **fig. 5** is reprinted in **fig. 12**. This c-net has no interior vertices, so it satisfies the 180° condition trivially. Thus it is semifoldable; the semifolded image is shown next to it in the diagram. However, as previously noted, there is no way to physically fold this piece of paper flat along the creases without cutting or self-intersection. A sufficiency condition for flat-foldability requires some additional constraints beyond semifoldability. We now need to consider the superposition ordering map σ.

Unlike the semifolding map, which is completely determined by the c-net up to isometry, there are often many superposition orderings that allow the paper to fold flat. Our goal is to determine whether there exists any ordering that makes this possible. To do this, we must first determine whether a given superposition ordering is possible.

Consider the x-ray view of the crane **(fig. 13)**. Each of the lines on this crane corresponds to some raw or folded edge, either visible on the top layer or hidden inside the model. The lines are the images of the c-edges under semifolding. Together they form a planar graph embedded on the crane silhouette. Each of the twelve faces of this graph represents a stack of paper layers, each layer having the same shape.

Definition 10. An *f-net* is the image of a semifoldable c-net C under μ. Specifically, it is the subset of $\mathbb{R}^2$ equal to $\mu(S)$ (where S is the sheet of paper), with the embedded graph $\mu(G)$ (where G is the crease pattern). The edges of this graph are *f-edges*, that meet at *f-vertices* and bound disjoint *f-faces*.

Imagine the crane was folded from a square of carbon-sensitive paper. Consider what would happen if we firmly rubbed this crane along every f-edge and then unfolded it. The carbon would leave a visible mark on every crease and raw edge, as well as any points inside the c-faces where a crease or raw edge happened to overlap. The result is shown in **fig. 14**. (The numbers will be explained momentarily.) This is the same as inverse-semifolding the f-net, so that each point on the paper that was marked by the carbon leaf becomes part of an edge or vertex.

Definition 11. An *s-net* is the inverse image of an f-net $\mu(C)$ under μ. It consists of the sheet of paper, S, with an embedded graph $\mu^{-1} \bullet \mu(G)$. This graph is a refinement of the crease pattern G; that is, $G \subset \mu^{-1} \bullet \mu(G)$. The faces, edges, and vertices of this graph are called *s-faces, s-edges,* and *s-vertices.*

Consider the significance of the s-net to our superposition ordering σ. Each s-face gets semifolded onto exactly one f-face, and every f-face is the image of at least one s-face. Thus the map σ should treat each s-face as a single unified entity; that is, σ acts as a constant function on each s-face. Now let F be an f-face, and let $\mu^{-1}(F)$ be the set of s-faces that get stacked up onto F when the paper is folded. The map σ should somehow number these s-faces to indicate the order in which they get stacked. If there are n s-faces in $\mu^{-1}(F)$, then σ should number them from 0 to $n-1$, inclusive. If the s-face X lies over the s-face Y in the folded model, we want $\sigma(X)$ to be greater than $\sigma(Y)$; if X lies underneath Y, we want $\sigma(X) < \sigma(Y)$.

Definition 12. Let C be a semifoldable c-net. A *superposition ordering* $\sigma : C \to \mathbb{Z}$ is any map that acts as a constant on each s-face and has the following property: For every f-face F, the n s-faces in the set $\mu^{-1}(F)$ are mapped to distinct integers between 0 and $n-1$, inclusive. (How σ affects the s-edges and s-vertices is inconsequential for determining flat-foldability, so it needn't be specified in this definition.)

The s-faces in any s-net can be partitioned into equivalence classes based on which f-face they each get mapped to. In **fig. 15**, the s-faces of a crane are assigned letters based on this partition. The lettered f-net beside it shows the correspondence. There are twelve s-faces labeled “G”, so σ maps them to the integers 0 through 11 based on their layering order in the folded crane. This numbering is shown in **fig. 14**.

10 THE NON-CROSSING CONDITION.

As the preceding paragraph described, a superposition order of the s-faces of a given model can be ascertained from the physical folded model itself. However, there exist superposition orderings that do not correspond to any flat origami model because the ordering would cause the paper to self-intersect along some s-edges. (In this case, saying the paper self-intersects refers to the physical reality of folding, rather than the mathematical abstraction we are building. In fact, any superposition ordering satisfying definition 12 will result in a one-to-one flat-folding map ω, but only a a few of these orderings are truly realizable with solid paper.) We need some additional axioms that specify when σ does not induce a self-intersection of the paper. Together, these axioms form the *non-crossing condition.* If we assume the paper is simply connected (so that the paper cannot be tied in knots) and the crease pattern is semifoldable, then the non-crossing condition is necessary and sufficient for flat-foldability.

Definition 13. If X and Y are adjacent s-faces separated by the s-edge E, and if E is contained in a crease, we say that E is an *s-crease* and we write $C(X, Y, E)$. If the s-faces are adjacent but E is not part of a crease, then we write $\sim C(X, Y, E)$.

Note that $\sim C(X, Y, E)$ implies that $\mu(X) \neq \mu(Y)$. Conversely, $C(X, Y, E)$ implies that $\mu(X) = \mu(Y)$, which in turn implies that $\sigma(X) \neq \sigma(Y)$.

Axiom 6. If $\sim C(X, Y, E)$ and $\sim C(X', Y', E')$ with $\mu(E) = \mu(E')$, we cannot have both $\sigma(X) > \sigma(X')$ and $\sigma(Y) > \sigma(Y')$. (This restriction prevents two c-faces from penetrating through one another's interiors. See **fig. 16a**.)

Axiom 7. If $C(X, X', E)$ and $\sim C(X'', Y, E')$ with $\mu(E) = \mu(E')$, we cannot have $\sigma(X) > \sigma(X'') > \sigma(X')$. (This restriction prevents a c-face from penetrating through a fold. See **fig. 16b**.)

Axiom 8. If $C(X, X', E)$ and $C(Y, Y', E')$ with $\mu(E) = \mu(E')$, we cannot have $\sigma(X) > \sigma(Y) > \sigma(X') > \sigma(Y')$. (This restriction prevents two folds from penetrating through one another. See **fig. 16c**.)

These three axioms exhaustively cover every possible circumstance in which the paper might self-intersect, as the diagram clearly shows. The site of the intersection must be along a pair of s-edges, of which neither, one, or both may be creases. If we choose the superposition order so that all three axioms are satisfied, then the non-crossing condition has been satisfied. With the following definition, we have finally solved our main problem.

Definition 14. A (simply connected) c-net is *flat-foldable* if and only if it is semifoldable and there exists a superposition ordering satisfying the non-crossing condition. In this case, the *flat-folding map* ω is the cartesian product of the semifolding and superposition ordering maps.

This "solution" to the problem of flat-foldability is less than satisfactory. We can confirm semifoldability by simply checking that the 180° condition holds at every c-vertex. If a c-net is semifoldable we can easily determine the unique semifolding map and use it to find the s-net. However, even having come that far, it is extremely difficult to know whether a valid superposition ordering exists. Trial and error seems to be the only effective way to determine this; if every single possible superposition ordering violates the non-crossing condition somewhere, then the c-net is not flat-foldable.

It is, in fact, computationally infeasible to use this method of trial and error. In 1996, Marshall Bern and Barry Hayes published their proof that the general problem of flat-foldability is NP-hard, meaning there exists no polynomial-time algorithm that determines the flat-foldability of an unsigned c-net. The problem of determining a valid superposition order for a c-net is also known to be NP-hard, even if the c-net is known beforehand to be flat-foldable.

There are no known easy-to-spot conditions that are equivalent to the rather inconvenient non-crossing condition. However, consideration of MV-assignments provides a good starting point for finding a valid superposition ordering, as we shall see in the next section.

11 MV-ASSIGNMENTS.

Recall from our discussion of crease patterns that all creases fall into one of two categories, mountain crease or valley crease. The difference is, when we make a mountain fold we swing part of the paper behind, whereas when we make a valley fold the paper swings in front. More precisely, let F_1 and F_2 be c-faces adjacent along a crease E, and let μ be a semifolding map that reflects F_2. Then E is a valley crease if F_2 gets placed on top of F_1, and it is a mountain crease if F_2 goes underneath. This suggests the next theorem.

Theorem 7. Let E_1 be the s-crease separating adjacent s-faces X and X', and let E_2 be the s-crease separating adjacent s-faces Y and Y'. Suppose E_1 and E_2 are both contained in the crease E, X and Y are contained in the c-face F, and X' and Y' are contained in the adjacent c-face F'. If $\sigma(X) > \sigma(X')$, then $\sigma(Y) > \sigma(Y')$.

Definition 15. Assume the same variable definitions as in the above theorem. Suppose $\sigma(X) > \sigma(X')$ and $\sigma(Y) > \sigma(Y')$. We say that E is a *valley crease* if μ acts on F as a reflection, and E is a *mountain crease* if μ acts on F' as a reflection.

Given a semifoldable c-net with a semifolding map μ and a superposition ordering σ (not necessarily in abidance with the non-crossing condition), we can can figure out which of the creases are mountains and which are valleys. In other words, applying a superposition ordering automatically "signs" the crease pattern.

We say that a signed c-net is *signed-flat-foldable* if it is possible to collapse the paper into a flat origami by making mountain and valley folds as indicated. Obviously, signed-flat-foldability is stronger than unsigned-flat-foldablity. A signed c-net may be signed-flat-foldable only if it is unsigned-flat-foldable. There exist a number of necessity conditions for signed-flat-foldability in addition to unsigned-flat-foldability. We can test the validity of a given superposition ordering of an unsigned c-net by checking the resultant MV-assignment against these conditions.

Notation. In the following theorem, the length of a line segment is denoted by absolute value bars.

Theorem 8. Suppose we have a signed-flat-foldable c-net containing faces F_1, F_2, and F_3, with F_2 adjacent to the other two. Let E_{12} and E_{23} be the edges separating F_2 from F_1 and F_3, respectively. If there exists a line L intersecting E_{12} and E_{23} such that $|L \cap F_2| < |R_{E_{12}}(L) \cap F_1|$ and $|L \cap F_2| < |R_{E_{23}}(L) \cap F_3|$, then E_{12} and E_{23} must have opposite MV-assignments. See **fig. 17**.

Corollary 9. Let F_1, F_2, and F_3 be consecutive faces in $F(V)$ for some c-vertex V. If A_1 and A_3 are both greater than A_2, then E_2 and E_3 must have opposite MV-assignments.

Corollary 9 can be used to demonstrate that **fig. 5** is not flat-foldable. The corollary's hypothesis applies at all three vertices, so no two of the creases can have the same MV-assignment. This is impossible.

Theorem 10 (Maekawa Theorem). Let V be an interior vertex in a signed-flat-foldable c-net. The number of mountain creases in $E(V)$ must differ from the number of valley creases in $E(V)$ by ± 2.

Again, none of these conditions are sufficient to prove the validity of a given superposition ordering, but they do help us by weeding out a large number of candidates. The problem of general flat-foldability is not completely solved, but the tools provided here are about as close to a full solution as we can hope to come.

12 SUMMARY AND CONCLUSION.

Our main goal in this study was to find a mathematical link between a crease pattern its corresponding origami model. We asked: Given an arbitrary unsigned crease pattern, is it possible to fold the paper along the creases so that the resulting model is flat? For the answer to be yes, the following conditions must all be met:

1. All crease lines must be straight line segments.
2. All interior vertices in the crease pattern must be of even degree.
3. At each interior vertex, the sum of every other angle must be 180°.
4. There must exist a superposition ordering function that does not violate the non-crossing condition.

When these conditions are all true, the crease pattern is flat-foldable by definition.

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SOURCES AND FURTHER READING.

belcastro, s-m. and Hull, T., "A mathematical model for non-flat origami", in: T. Hull ed., *Origami*^{3}, (2002) 39-51.

Bern, M. and Hayes, B., "The complexity of flat origami", *Proceedings of the 7th Annual ACM-SIAM Symposium on Discrete Algorithms*, (1996) 175-183.

Frigerio, E., "Origami geometry: old and new", in: H. Huzita ed., *Proceedings of the First International Meeting of Origami Science and Technology*, Ferrara, (1989) 379-386.

Hull, T., "On the mathematics of flat origamis", *Congressus Numerantium*, 100 (1994) 215-224.

Hull, T., "The combinatorics of flat folds: a survey", in: T. Hull ed., *Origami*^{3}, (2002) 29-38.

Justin, J., "Towards a mathematical theory of origami", in: K.Miura ed., *Origami Science and Art: Proceedings of the Second International Meeting of Origami and Scientific Origami*, Seian University of Art and Design, Otsu, (1997) 15-29.

Kawasaki, T., "On the relation between mountain-creases and valley-creases of a flat origami" (abridged English translation), in: H. Huzita ed., *Proceedings of the First International Meeting of Origami Science and Technology*, Ferrara, (1989) 229-237.

Kawasaki, T., "$R(\gamma) = I$", in: K.Miura ed., *Origami Science and Art: Proceedings of the Second International Meeting of Origami and Scientific Origami*, Seian University of Art and Design, Otsu, (1997) 31-40.

Maekawa, J., "Similarity in origami", in: K.Miura ed., *Origami Science and Art: Proceedings of the Second International Meeting of Origami and Scientific Origami*, Seian University of Art and Design, Otsu, (1997) 109-118.