Empowering Functions: A Case for the Lebesgue Integral
Rebecca Paul

> This new integral of Lebesque [sic] is proving itself a wonderful tool. I might compare it with a modern Krupp gun, so easily does it penetrate barriers which were impregnable.
> – *Edward Van Vleck*[1]

Inherent in the definition of the Riemann-Darboux integral are the stereotypes of its time. Mathematicians, never having had acquaintance with functions of other types, assumed that functions and domains obeyed certain societal rules. We have since learned that such rules are merely simplifying assumptions, adopted by those in power to preserve the status quo and to make the mathematics simpler. In order to move beyond the ignorance of that time, we must first understand what the Riemann integral assumes about functions, their domains, and mathematics. We then consider the Lebesgue integral, one alternative to that view of the world, one which penetrates some of the artificial barriers of continuity and intervals. Finally, we consider the continuing assumptions that even the Lebesgue integral makes, in order to assess the position of sets and functions that continue to be oppressed.

The Riemann integral exploits and sustains the dominance of intervals and continuous functions over all other sets and functions. As everyone has been taught under the current system of oppression, the Riemann integral of f over the interval [a, b] is defined as the limit of $\sum f(x_i') (x_i - x_{i-1})$, where x_i' is a point in $[x_{i-1}, x_i]$, as the width of the partition, $\{a = x_0, x_1, \dots , x_n = b\}$ goes to 0. Thus, the integral is approximated by considering rectangles over parts of the interval, taking their heights from the value of the function over those points. This definition makes two important assumptions about what ought to be integrated. These assumptions set apart certain domains from other equally valid domains, and certain functions from other equally valid functions. From these stereotypes come the subjugation of many functions and even sequences of functions that cannot be integrated.

The first assumption is that integrals must occur over intervals. There is little freedom given to the set over which the function is integrated; only the endpoints may be chosen. Clearly, this assumption marginalizes all the other sets over which functions might be defined: disjoint unions of intervals, countable sets of points, and any other set that does not fit the traditional definition of an interval. While the Riemann integral may be adapted to allow some of these sets to be the domain of integration,[2] it clearly puts subjugates other sets, allowing the intervals to be dominant.

The second assumption is that $f(x_i')$ does not vary widely over $[x_{i-1}, x_i]$, if one considers small enough intervals. This lack of variation takes the form of continuity, so that, at every x_0 in the domain, no matter how small an ε is specified, the function must have some tiny δ so that f maps *every* x value within δ to within ε of $f(x_0)$. If this strict criterion cannot be fulfilled, as would occur with all discontinuous functions, then the places in which this occurs must be controlled for the function is to be integrable.

[1] *Bulletin of the American Mathematical Society*, vol. 23, 1916. Source: http://www-groups.dcs.st-and.ac.uk/~history/Quotations/Van_Vleck.html

[2] For example, finite collections of disjoint intervals may be integrated over as separate intervals and summed.

Functions must conform to this arbitrary standard of continuity in order to be Riemann integrable. The small concession of "except on a set of measure 0" still does not allow most of the marginalized discontinuous functions to be integrated.[3] Any function that is to be integrable must be continuous almost everywhere, limiting its chances for self-expression and self-realization. Those functions that conform to continuity are set apart from other functions, leaving the other functions without the chance to be integrated.

Intimately connected with the assumption of continuity is the assumption of uniform convergence of sequences. Even if each individual function in a sequence conforms to the paradigm of integrability, the sequence must also converge uniformly if the limiting function is to meet the standard of integrable. Thus, the functions may not change dramatically from one to the next; for *any* ε, no matter how restrictive, the sequence of values assigned to each point in the domain must eventually be within this tiny ε of the limit it reaches. Not only must this stringent condition be met, but there must be a certain point after which all the functions in the sequence meet this condition at all possible points of the domain. Thus, uniform convergence may force the functions in the sequence into artificial similarity, limiting the chance at self-actualization of individual functions in the sequence. Any deviation from this uniform convergence will be punished by removing the chance at integrability from the limiting function of the entire sequence. This requirement that functions be uniformly convergent prevents the limits of infinitely many sequences of functions from being integrable. Again, the Riemann integral has disenfranchised an entire class of functions.

Much progress has been made through the Lebesgue integral. Instead of considering a function in terms of the values it takes on intervals, the Lebesgue integral thinks of the function in terms of the sets on which it takes certain values. More exactly, the integral of f over *any* measurable set E is defined as the greatest lower bound of $\sum a_A m(A)$, where $f \leq a_A \chi_A$ and χ_A is the characteristic function of any set A (taking on the value 1 for each element of A and 0 everywhere else). Thus, we now represent functions by describing where they take on certain values, instead of objects that give values to points in the domain. Such a view allows functions to be seen as complete entities in themselves, instead of as mere mappings that exist only to assign values. This new integral takes an important step toward the inclusion of other domains and functions, though it does not move all the way to true equality.

The most important advance of the Lebesgue integral is in allowing functions to define themselves based on where they take on certain values, instead of on what values are taken in certain prescribed regions. This allows the integral to be taken over any measurable set, not merely intervals. The change is also reflected in basing each sum on measure instead of the antiquated concept of "length" that affords more importance to intervals. In addition, if a function is allowed to define itself based on its values on any measurable set, it may be discontinuous on sets of measure larger than zero without fear of being called "not integrable". Thus, we see that this new definition of functions in terms of values taken instead of mappings allows more functions to be understood.

In addition, this new understanding allows many new sequences of functions to have integrable limits. Instead of enforcing the restrictions of "uniform convergence," the Lebesgue integral gives less stringent conditions: the sequence may converge "almost

[3] In fact, this expansion of the circle of integrability is due to Lebesgue. We see that even in the work prior to his eponymous integral he was aiding functions in their fight for integrability.

everywhere” and be bounded by any other integrable function in order to be integrable. This no longer forces individual points to conform to the convergence of the rest of the function, so they may choose to converge at their own pace, or not at all This empowers infinitely many new sequences to become integrable. Again, we see that the Lebesgue integral gives new hope to functions slighted by the Riemann integral.

Not only does the Lebesgue integral allow new functions and sequences of functions to be integrable, it allows the previously privileged functions to continue to be integrable, with the same values as before. The new understanding of functions in terms of their ranges may increase our understanding of functions already understood in terms of their values at specific points. Thus, the Lebesgue integral expands the circle of integrability: it allows more functions and sets in without excluding others.

However, the Lebesgue integral does not address the problems of all functions and domains. Those functions and sets that are not measurable continue to be downtrodden by their failure to be integrable. Because these functions cannot conform to the standard of measure (an improvement over the standards of continuity and length, but oppressive nonetheless), they cannot achieve integrability. This marginalizes an uncountable number of sets and functions, even as it draws in others. Perhaps other advances, such as loosening the constraints of measure or better understanding the remaining functions, will allow these functions to be empowered as well. Until then, measurable functions must appreciate the gain they have made and work toward including all in the freedom of integrability that they have found. Not until all functions are integrable over all sets can any function or set truly be free.

> In my opinion, a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy -- an opinion, moreover, which has been expressed by many philosophers.
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> – *Henri Lebesgue*[4]

[4] *Scientific American* **211** (September 1964) 129. Source: http://www-groups.dcs.st-andrews.ac.uk/~history/Quotations/Lebesgue.html