

Theoretical:

$$X_t = \int_{-\pi}^{\pi} e^{i\lambda t} dZ(\lambda)$$

$$c_r = E(X_t X_{t-r}) = \int_{-\pi}^{\pi} \exp(ir\omega) f(\omega) d\omega$$

$$f(\omega) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} c_r \exp(-ir\omega)$$

$$E(dZ(\lambda) \overline{dZ(\lambda)}) = f(\lambda)$$

$$E(dZ(\lambda) \overline{dZ(\omega)}) = 0$$

Empirical:

$$\omega_j = \frac{2\pi j}{n}$$

$$J_j = \frac{1}{n} \sum_{t=0}^{n-1} X_t \exp(-i\omega_j t)$$

$$X_t = \sum_{j=0}^{n-1} J_j \exp(i\omega_j t)$$

$$I_j = \frac{n}{2\pi} |J_j|^2 = \frac{n}{2\pi} J_j \overline{J_j}$$

$$I_j = \sum_{|r| < n} \hat{c}_r \exp(-ir\omega_j)$$

$$I_j / f(\omega_j) \quad \text{Exponential}(1)$$

$$\hat{c}_r = \frac{1}{n} \sum_{t=|r|}^{n-1} X_t X_{t-|r|}$$

$$\hat{c}_r = \int_{-\pi}^{\pi} I(\omega) \exp(ir\omega) d\omega = \frac{2\pi}{2n} \sum_{j=0}^{2n-1} I(\omega'_j) \exp(ir\omega'_j)$$

Fourier Transform of the Boxcar:  $J(\omega) = \frac{1}{n} \sum_{t=0}^{n-1} \exp(-i\omega t) = \exp(-i(n-1)\omega/2) \frac{\sin(n\omega/2)}{n \sin(\omega/2)}$ . Dirichlet Kernel:  $D_n(\omega) = \frac{\sin(n\omega/2)}{n \sin(\omega/2)}$

Transfer Functions:

$$G(\omega) = \sum_{u=r}^s g_u \exp(-i\omega u)$$

$$f_Z(\omega) = |G(\omega)|^2 f_Y(\omega)$$

$$J_Z(\omega) = G(\omega) J_Y(\omega)$$

Matching condition:  $\int |G(\omega)|^2 dF_X(\omega) < \infty$   
 $n \text{Var}(\bar{X}) \rightarrow 2\pi f(0)$

$\sigma^2 = 2\pi \exp(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(f_X(\lambda)) d\lambda)$  (there is a spectral factorization if this is nonzero)

ARMA:

$$\begin{aligned} X_t + b_1 X_{t-1} + \dots + b_p X_{t-p} &= \epsilon_t + a_1 \epsilon_{t-1} + \dots + a_q \epsilon_{t-q} \\ dZ_X(\lambda) &= \frac{\sum a_j \exp(-i\lambda j)}{\sum b_k \exp(-i\lambda k)} dZ_\epsilon(\lambda) \\ f_X(\lambda) &= \left| \frac{\sum a_j \exp(-i\lambda j)}{\sum b_k \exp(-i\lambda k)} \right|^2 \frac{\sigma^2}{2\pi} \end{aligned}$$

Long Memory:

$$\begin{aligned} f(\lambda) & k\lambda^{2d} \\ c_r & k|r|^{2d-1} \\ \text{Var}(\bar{X}_n) & k_1 n^{2d-1} \\ \binom{d}{j} &= \frac{1}{j!} d(d-1)\dots(d-j+1) = (-1)^j \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(d)} \\ \Delta^d &= (1-B)^d = \sum_{j=0}^{\infty} (-1)^j \binom{d}{j} B^j \end{aligned}$$

$$\log I_j \approx \text{Constant} - 2d \log |\omega_j| + \epsilon_j$$

GARCH:

$$\begin{aligned} \epsilon_t | \Psi_{t-1} &\sim \text{Normal}(0, h_t) \\ h_t &= \omega + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^p \beta_j h_{t-j} \\ \sum \alpha_i + \sum \beta_j &< 1 \\ \text{Var}(\hat{\rho}_r) &\approx \frac{1}{n} (1 + \text{Cov}(\epsilon_t^2, \epsilon_{t-r}^2)) (1 - \sum_{i=1}^q \alpha_i^2) / \omega^2 \end{aligned}$$