Microeconomics Summary

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1 Mathematical Preliminaries

1.1 Solving Optimization Problems

Definition Suppose $f(p, w)$ is a well-defined function. Suppose $(p^q, w^q) \rightarrow (p^0, w^0)$ is a convergent sequence. $f(p, w)$ is *continuous* if $f(p^q, w^q) \rightarrow f(p^0, w^0)$.

Theorem 1.1 Weierstrass. *A real-valued continuous function on a non-empty compact set achieves a maximum and a minimum. That is, if K is compact, $f : K \rightarrow R$, there exists $x^* \in K$ such that $f(x^*) \leq f(x)$ for all $x \in K$ and a point $x^{**} \in K$ such that $f(x) \leq f(x^{**})$ for all $x \in K$. Note that x^*, x^{**} need not be unique.*

Any closed and bounded set in $R^n$is compact.

In some problems, f may be defined on set, D, that is not compact. However, we may be able to find a compact set, $K \subset D$, such that every point in K is larger (smaller, if we're looking for a minimum) than every point in $D - K$. Then, a maximum must exist in K, and it must be the maximum for all of D as well.

In the *standard consumer theory problem*, we maximize a utility function, $u : R^l_+ \rightarrow R$, subject to $p \cdot x \leq w$. The budget set will be compact if and only if every element of p is positive. If the budget set is compact, $w \geq 0$, and u is continuous, then the Weierstrass theorem can be applied. The solution of the problem gives the *demand correspondence* (since the point that maximizes u need not be unique). This also yields the *indirect utility function*, $v(p, w) = u(x^*)$.

Profit maximization chooses the amount of output, y, to maximize $qy - C(p, y)$, where q is the price of output, and $C(p, y)$ is the cost of producing y units at input prices p. (Equivalently, we maximize $qy - p \cdot x$, subject to $f(x) \geq y$, where x is the amount of input.) In this case, y need not be bounded above, so there might not be an optimum. Also, if an input x is free, there might be no optimum. (If the set of feasible inputs and outputs in compact, however, we may be able to apply Weierstrass.)

Definition Let $\xi(p, w)$ be a demand correspondence. We define the *graph* of ξ by $G = \{(x, p, w) : x \in \xi(p, w)\}$. If the graph is closed, then we say that the correspondence has the *closed graph property.*

The closed graph property only considers sequences that converge. (Sequences that don't converge include some cases where $p^q \to 0$.)

One potential problem is the *Arrow corner*. Suppose $\frac{w}{p_1}$ and $p_2 > 0$ are constant and $p_1 \downarrow 0$. In the limit, $p_1 = w = 0$. For small enough p_1, the optimal choice will be fixed at $(\frac{w}{p_1}, 0)$. However, in the limit, the entire x-axis is feasible, and there is no optimum. Therefore, we insist that $w_0 > 0$.

Theorem 1.2 *The demand correspondence, ξ, has the closed graph property. That is, suppose $(p^q, w^q) \to (p^0, w^0)$, $x^q \in \xi(p^q, w^q)$, $x^q \to x^0$, and $w_0 > 0$. Then, $x^0 \in \xi(p^0, w^0)$.*

Proof Since $p^q x^q \leq w^q$, $p^0 x^0 \leq w^0$. Since $x^q \in R^l_+$, $x^0 \in R^l_+$ as well. Thus, x_0 is in the budget set defined by (p^0, w^0). Suppose x^0 is not the choice from the budget set. Then, there is some $\overline{x} \in B(p^0, w^0)$ with $u(\overline{x}) > u(x^0)$. Suppose $\overline{x} \cdot p^0 = w^0 > 0$. Then, for all $\lambda < 1$, $p^0(\lambda \overline{x}) < x^0$. By continuity, we may choose λ close to 1 such that $u(x_0) < u(\lambda \overline{x})$ and $p\lambda\overline{x} < w^0$. (If $\overline{x}p^0 = 0$, then $\lambda = 1$ works for this.) For large enough q, $p^q(\lambda\overline{x}) \leq w^q$. Since $u(x^q) \to u(\overline{x})$, we have $u(\lambda\overline{x}) > u(x^q)$. This contradicts the optimality of x^q in its budget set, and we have a contradiction. ∎

Similar proofs show that the indirect utility function and profit function are continuous.

Theorem 1.3 *Suppose $f : S \to R$. If f is convex and S is convex, then every local minimum is a global minimum in S. If f is concave and S is convex, then every local maximum is a global maximum in S. The set of minima (or maxima) is a convex set in S.*

Note that quasi-concavity is not enough.

This means that finding points that satisfy the first order conditions is sufficient in these cases.

Definition Let $f : R^n \to R$ be concave. The *gradient inequality* states that $f(y) \leq f(x) + \nabla f(x)(y - x)$. That is, the linear approximation of a concave function always lies above it. (The reverse holds for convex functions.)

Definition Let $g_i : R^n \to R$ for $i = 1, ..., r$. Consider the constraints $g_i(x) \leq 0$ for $i = 1, ..., r$. The *constraint qualification* states that at any feasible point, x, there is a path $\phi : [0, 1] \to R^n$, such that $\phi(0) = x$, $g_i(\phi(t)) < 0$ for all $i = 1, ..., r$ and $t > 0$, and ϕ is differentiable. That is, we may find a differentiable path from any point into the interior.

Theorem 1.4 Kuhn-Tucker. *Let $f : R^n \to R$, $g_i : R^n \to R$ for $i = 1, ..., r$. Suppose we want to minimize f subject to the constraints, $g_i(x) \leq 0$ for $i = 1, ..., r$. Assume that f, g_i are continuously differentiable everywhere and that the constraint qualification holds.*

The necessary conditions for x^* to solve this problem are that there exists multipliers $\lambda^* \geq 0$ such that:

$$\nabla f(x^*) + \sum_{i=1}^{r} \lambda_i^* \nabla g_i(x^*) = 0$$

(this is the first order condition) and

$$\lambda_i^* g_i^*(x^*) = 0$$

for $i = 1, ..., r$ (these are the complementary slackness conditions*, and ensure that any positive multiplier has a binding constraint and that any non-binding constraint has a zero multiplier).*

Suppose $f, g_1, ..., g_r$ are convex and continuously differentiable. If x^* satisfies the first order condition and the complementary slackness conditions above for some $\lambda^* \geq 0$, then x^* is a minimum.

1.2 Hyperplanes Theorems

Theorem 1.5 Separating Hyperplanes Theorem. *Let S be a non-empty convex set. Let $x \notin S$. Then, there exists a vector $p \neq 0$ such that $p \cdot x < \alpha < p \cdot y$ for all $y \in S$.*

Proof Let $y_0 \in S$. Define $S' = \{y : \|x - y\| \leq \|x - y_0\|\} \cap S$. This is a compact set. Since $\|x - y\|$ is a continuous function on S', there is a minimum, y^*. Set $p = y^* - x$. Then, p is orthogonal to y^*. Let $z \in S$, $z \neq y^*$. By the convexity of S, $(1 - t)z + ty \in S$ for all $t \in [0, 1]$. Define:

$$\begin{aligned} f(t) &= \|tz + (1 - t)y - x\| \geq \|y - x\| \\ f'(t) &= \frac{d}{dt}(\sum_{h=1}^{l}(tz_h + (1 - t)y_h^* - x_h)^2)^{1/2} \\ f'(0) &= \frac{1}{2}\sum_{h=1}^{l} 2(y_h^* - x_h)(z_h - x_h)\frac{1}{\|y^* - x\|} \geq 0 \end{aligned}$$

(FIGURE THIS OUT) Thus, $p \cdot (z - y^*) \geq 0$, and $p \cdot z \geq p \cdot y^*$. Furthermore, $p \cdot (y - x) = (y - x) \cdot (y - x) > 0$. Thus, $p \cdot z \geq p \cdot y > p \cdot x$. We set $\alpha = \frac{1}{2} p \cdot (x + y)$, and the supporting hyperplane is $H_\alpha(p) = \{z | p \cdot z = \alpha\}$. ∎

Corollary 1.6 Supporting Hyperplanes Theorem. *Let S be a non-empty, convex set. Let x be on the boundary of S, but not in S. Then, there exists $p \neq 0$ such that $p \cdot y > p \cdot x$ for all $y \in S$.*

Proof *Sketch.* Let S be given with $x \in \overline{S} \backslash S$. Choose z_n outside S and let $z \to x$. Then, in the limit, every point, y, in the interior will have $p \cdot y > p \cdot x$. ∎

1.3 Fixed Point and Uniqueness Theorems

Theorem 1.7 Brouwer's Fixed Point Theorem. *If $K \subset R^n$ is a non-empty, convex, compact set and $f : K \to K$ is continuous, then there exists some $x^* \in K$ such that $f(x^*) = x^*$.*

Theorem 1.8 Kakutani. *Suppose K is a compact, convex, non-empty set in R^n and $\phi : K \to K$ is a non-empty, convex-valued correspondence with a closed graph. Then, there is some $x^* \in K$ with $x^* \in \phi(x^*)$.*

Definition Suppose we have $f : U \to R^n$, where U is an open set. Suppose there is some $x \in U$ with $f(x) = 0$. We say that x is *locally unique* if there is a neighborhood, V, of x such that if $y \in V$ and $y \neq x$, then $f(y) \neq 0$.

Theorem 1.9 Inverse Function Theorem. *Suppose $U \subset R^n$ is open and $f : U \to R^n$ is r times differentiable at x. Suppose $\nabla f(x)$ is non-singular. Then, there exist a neighborhood of x, $V \subset R^n$, and a C^r function, $f^{-1} : V \to R^n$, such that $f \cdot f^{-1}(y) = y$ for all $y \in V$. In addition, $(\nabla f^{-1})(f(x)) = (\nabla f(x))^{-1}$.*

Proposition 1.10 *If $f(x) = 0$ and the conditions of the inverse function theorem hold (with $r \geq 1$), then $f(x)$ is locally unique.*

Theorem 1.11 Transversality Theorem. *Suppose $f : R^{m+p} \to R^n$ is continuously differentiable and $f(x) = 0$. If the $n \times (m + p)$ matrix $\nabla f(x; q)$ has rank n whenever $f(x; q) = 0$, then for almost every q (ie., except on a set of measure 0), the $n \times m$ matrix $\nabla_x f(x; q)$ has rank n when $f(x; q) = 0$.*

2 Walrasian Equilibrium

Definition A *commodity* is a physical object together with its location, timing (date of delivery), and contingencies (based on exogenous uncertainty). This allows price to depend on more than the physical attributes of the good.

Definition We have *complete markets* if any possible commodity can be traded, and therefore there is a known price for every commodity.

Given a finite set of commodities, $h = 1, ..., l$, the commodity space is R^l, and an element, x, of the commodity space is called *bundle*. This is a vector space, since we may add bundles and multiply bundles by scalars. We define *prices*, p, as elements of R^l (they may be negative). Note that only relative prices matter, so αp and p will lead to the same decisions. The *value* of a bundle, given certain prices, is $p \cdot x = \sum_{h=1}^{l} p_h x_h$. This is a bilinear map (since it is linear in both p and x).

Implicit in this models are:

- Perfect Competition: All agents are price-takers.
- No externalities: The consumption of one agent does not affect the utility of another agent.

- Symmetric information: There is no adverse selection, moral hazard, or question about quality.
- Complete markets: Everything is traded at one time at the beginning of time, and after that are deliveries only (with perfect enforcement).

Violations will lead to inefficiency in most cases.

2.1 Pure Exchange Economies

Definition In a *pure exchange economy*, there is:

- A commodity space, R^l_+.
- A collection of individuals, $i = 1, ..., I$, each with a consumption set, X_i, and a utility function, $u_i(x)$, that is continuously differentiable on an open set containing the consumption set, strictly increasing, and weakly concave.
- An initial endowment for each individual, $e_i \in X_i$ (This ensures that the individual could still consume in the consumption set, even without trade.)

We assume that the total endowment, $\sum_{i=1}^I e_i$, is strictly positive for each good.

Definition An *allocation* is $x = \{x_i\}_{i=1}^I$, which specifies a bundle for each consumer, subject to the restriction that each consumer's bundle belongs to that consumer's consumption set. An allocation is *attainable* if it uses the total endowment. That is $\sum_{i=1}^I x_i = \sum_{i=1}^I e_i$.

Definition An attainable allocation is *strongly Pareto efficient (or strongly Pareto optimal)* if there is not another attainable allocation such that each consumer i is at least as well off and some i is strictly better off. An attainable allocation is *weakly Pareto efficient* if it is impossible to make every agent strictly better off.

Note that strong Pareto efficiency implies weak Pareto efficiency.

Definition The *utility possibility set*, $\mathcal{U}$, is given by $\mathcal{U} = \{(u_1, ..., u_I) : u_i \leq u_i(x_i)$ for some attainable allocation $x\}$.

Proposition 2.1 *The utility possibility set is convex.*

Proof Suppose $u, u' \in \mathcal{U}$. Then, there exist feasible allocations x, x' such that $u_i \leq u_i(x_i)$ and $u'_i \leq u_i(x'_i)$ for all i. Let $t \in (0, 1)$. Then, for each i,

$$\begin{aligned} tu_i + (1-t)u'_i &\leq tu_i(x_i) + (1-t)u_i(x'_i) \\ &\leq u_i(tx_i + (1-t)x'_i) \end{aligned}$$

by the concavity of each u_i. Furthermore, the allocation $tx_i + (1-t)x'_i$ for each i is attainable, since $t\sum_{i=1}^I x_i + (1-t)\sum_{i=1}^I x'_i = \sum_{i=1}^I e_i$. ∎

Any efficient allocation lies on the frontier of the utility possibility set. Furthermore, for any $x \in Boundary(\mathcal{U})$, there is a supporting hyperplane (which is tangent at x and lies outside $\mathcal{U}$) by the concavity of $\mathcal{U}$. We may write the hyperplane as $\sum \alpha_i u_i = k$. This leads to the maximization problem of maximizing $\sum \alpha_i u_i$ subject to $(u_1, ..., u_I) \in \mathcal{U}$. For any $\alpha_i > 0$, a point is a solution to this problem if and only if it corresponds to a Pareto efficient outcome.

Theorem 2.2 Negishi. *All Pareto efficient points can be represented as solutions to the problem of maximizing* $\sum_{i=1}^{I} \alpha_i u_i(x_i)$ *subject to* $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i$, *with* $x_i \geq 0$ *for* $i = 1, ..., I$.

This proof depends on the convexity of $\mathcal{U}$, which implies that each u_i must be concave; otherwise, there is no supporting hyperplane for some points.

In general, we cannot separate welfare from distribution; if α_i is larger, than α_i will necessarily have higher welfare (and therefore more goods); this may affect the overall distribution of goods dramatically.

Definition A utility function, $u_i : R^l_+ \to R$, is *locally non-satiable* at x_0 if for any $\epsilon > 0$, there exists x' with all elements strictly greater than x_0 such that $u_i(x') > u_i(x_0)$ and $|x' - x_0| < \epsilon$.

(This rules out "thick" indifference curves, where there are bands of constant utility.)

Proposition 2.3 *If preferences are locally non-satiable, then weak Pareto efficiency implies strong Pareto efficiency.*

Proof Suppose x is a point that is weakly Pareto efficient but not strongly Pareto efficient. Then, there is some attainable x' such that $u_i(x'_i) \geq u_i(x_i)$ for all i, with strict inequality for some j. Then, for that j, $p \cdot x'_j > p \cdot x_j$. Suppose that $p \cdot x'_i < p \cdot x_i$ for any i. Then, there is some x''_i such that x''_i is strictly preferred to x''_i to x'_i in any neighborhood by local non-satiability. Then, $p \cdot x''_i < p \cdot x_i$ and $u_i(x''_i) > u_i(x'_i)$ as well. This contradicts the fact that we are at equilibrium. Thus, $p \cdot x'_i \geq p \cdot x_i$ and the inequality is strict for x_j. Adding up both sides over all i would violate attainability. ∎

Definition Suppose we have an exchange economy, $\mathcal{E} = \{(R^l_+, u_i, e_i)\}_{i=1}^{I}$, with $e_i \in R^l_+$. A *Walrasian (competitive) equilibrium* consists of an attainable allocation, x^*, and a price vector, $p^* \neq 0$, such that for all $i = 1, ..., I$, x^*_i solves the consumer's problem given the prices. That is, x^*_i maximizes u_i subject that $x_i \in R^l_+$ and $p^*_i x_i \leq p^*_i e_i$.

In equilibrium, when each consumer maximizes utility subject to the budget constraint, the markets clear.

Theorem 2.4 *Every Walrasian equilibrium is (weakly) Pareto efficient.*

Proof Suppose (x^*, p^*) is a Walrasian equilibrium and not weakly Pareto efficient. Then, there exists an attainable x with $u_i(x_i) > u_i(x_i^*)$ for all i. Because the consumer did not optimize with this bundle, it must not be in the budget set, and $p^* \cdot x_i > p^* \cdot e_i$. This is true for all i. Then, $\sum_{i=1}^{I} p^* \cdot x_i > \sum_{i=1}^{I} p^* \cdot e_i$. This contradicts the attainability of x, since we must have $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i$. $\blacksquare$

Definition An agent has a *direct revealed preference* for x_1 over x_2 if both x_1 and x_2 are in the budget set (feasible) and the consumer chooses x_1 over x_2. This defines a binary relation, $x_1 R^D x_2$. The transitive closure of this binary relation defines *indirectly revealed preferences*; that is, x_1 is *indirectly revealed preferred* to x_k, $x_1 R x_k$, if there is a sequence of directly revealed preferences, $x_1 R^D x_2 R^D ... R^D x_k$.

By the strong axiom of revealed preferences, if $x_1 R x_k$ and x_k is chosen, then x_1 must not be in the budget set.

Theorem 2.5 First Fundamental Theorem of Welfare Economics. *Suppose (x^*, p^*) is a Walrasian equilibrium and every u_i satisfies local non-satiability. Then, x^* is strongly Pareto efficient.*

Proof Suppose not. Then, there exists an attainable x with $u_i(x_i) \geq u_i(x_i^*)$ for all i with strict inequality for at least one i. Suppose $p^* \cdot x_i < p^* \cdot e_i$. By local non-satiability, we could increase the consumption of each good by some small ϵ to x_i', such that $p^* \cdot x_i' \leq p^* \cdot e_i$ and $u_i(x_i') > u(x_i) \geq u_i(x_i^*)$. This contradicts the optimality of x_i. Thus, $p^* \cdot x_i \geq p^* \cdot e_i$ for all i, and $p^* \cdot x_i^* > p^* \cdot e_i$ when $u(x_i) > u(x_i^*)$. Summing over i shows that $\sum_{i=1}^{I} p^* \cdot x_i > \sum_{i=1}^{I} p^* \cdot e_i$, which contradicts feasibility. $\blacksquare$

At equilibrium (assuming it is not at a corner solution), the marginal rates of substitution are equal across consumers.

Definition An attainable allocation, x^*, can be *decentralized* if there exists a price vector $p \neq 0$ such that for all i, $x_i \in X_i$, if $u_i(x_i) > u_i(x_i^*)$ then $p \cdot x_i > p \cdot x_i^*$.

Definition An *equilibrium with lump sum transfers* consists of an attainable allocation, x^*, a price vector, $p^* \neq 0$, and transfers, $t^* = (t_1^*, ..., t_I^*)$, such that $\sum_{i=1}^{I} t_i^* = 0$, and for any i, x_i^* maximizes $u_i(x_i)$ subject to the budget constraint, $p^* \cdot x_i^* \leq p^* \cdot e_i + t_i^*$, $x_i \in X_i$.

Note that $t_i = p \cdot x_i^* - p \cdot e_i$, and $\sum_{i=1}^{I} t_i = p \cdot \sum_{i=1}^{I} (x_i^* - e_i) = 0$.

Proposition 2.6 *An attainable allocation, x^*, can be decentralized if and only if there exists an equilibrium with lump sum transfers that has allocation x^*.*

Definition Let x_i be a point in a consumer's consumption set; let u_i be the consumer's utility function. The *preferred set* is given by $P_i(x_i) = \{y \in R_+^l | u_i(y_i) > u_i(x_i)\}$.

Lemma 2.7 *Let X_i be open. Let u_i be locally non-satiable and quasi-concave. Let x^* be a weakly Pareto efficient allocation. Then, $P_i(X_i^*)$ is non-empty, open, and convex.*

Proof By the continuity of u_i and the fact that X_i is open, we may find a small open ball about any point contained in $P_i(x_i^*)$. Since preferences are locally non-satiable, there is a point preferred to x_i^*, which is in $P_i(x_i^*)$. Convexity follows from quasi-concavity. ∎

Theorem 2.8 Second Fundamental Theorem of Welfare Economics. *Suppose each consumption set, X_i, is open and convex, and each utility function, u_i, is continuous, quasi-concave, and locally non-satiable. Then any weakly Pareto efficient allocation can be decentralized (and therefore it can be an equilibrium with lump sum transfers).*

Proof Define $Z_i = P_i(x_i^*) - \{x_i^*\}$. This is the set of vectors from x_i^* into the preferred set; equivalently, Z_i shifts $P_i(x_i^*)$ so that x_i^* is at the origin. Define $Z = \sum_{i=1}^I Z_i = \{z : \sum_{i=1}^I z_i, z_i \in Z_i\}$. Since each z_i would make each individual better off, any $z \in Z$ is the amount of resources needed to make every individual better off. Note that each Z_i and therefore Z is non-empty, open, and convex, because these properties are preserved by set addition.

Note that $0 \notin Z$, because if there were $\sum_{i=1}^I z_i = 0$, and $u_i(x_i^* + z_i) > u_i(x_i^*)$, this would contradict weak Pareto efficiency, since it is a way to make everyone better off with no additional resources. However, $0 \in \overline{Z}$ by local non-satiability, since we may find a sequence $z_i^n \to 0$ such that $u_i(x_i^* + z_i^n) > u_i(x_i^*)$ and $z_n = \sum_{i=1}^I z_i^n \to 0$ as well. Thus, by the supporting hyperplanes theorem, there exists p^* such that $p^* \cdot z > 0$ for all $z \in Z$. That is, any bundle that would make everyone better off would have a strictly positive value.

We show that $p^* \cdot z_i > 0$ for all $z_i \in Z_i$. Suppose there exists j such that $p^* \cdot z_j \leq 0$. Since Z_j is open, we may choose $z_j \in Z_j$ such that $p^* \cdot z_j < 0$. For all $i \neq j$, there exists $z_i^n \to 0$ in Z_i. Define $z^n = z_j + \sum_{i \neq j} z_i^n$. Then, $z^n \in Z$ for all n. This means that $p^* \cdot z^n > 0$. However, $z^n \to z_j$, which means that $\lim_{n\to\infty}(p^* \cdot z^n) = p^* \cdot z_j < 0$. This is a contradiction, and we must have $p^* \cdot z_i > 0$ for all i.

Thus, p^* decentralizes the allocation x^*, and therefore this allocation can be an equilibrium with lump sum transfers. ∎

Thus, it is enough to transfer income and vary prices in order to find an efficient allocation; the actual goods do not need to be redistributed. However, this may fail when x_i^* is not in the interior.

In general to support an efficient allocation, set $p = \nabla u(x_i)$ for any x_i which does not lie on the boundary.

Definition A *quasi-equilibrium* is (x^*, p^*) where x^* is an attainable allocation, $p^* \neq 0$, such that for any i, if $u_i(x_i) > u_i(x_i^*)$ then $p^* x_i \geq p^* x_i^*$.

Proposition 2.9 *Suppose $p^* \cdot x_i^* > \inf(p^* \cdot x_i)$ (that is, x_i^* is not the cheapest bundle in the consumption set), X_i is convex, and u_i is continuous. Then, a quasi-equilibrium, x^* is decentralizable.*

Proof Suppose there exists x_i such that $u_i(x_i) > u_i(x_i^*)$ and $p^* \cdot x_i = p^* \cdot x_i^*$. Let $\overline{x_i}$ be a point which costs $\inf(p^* \cdot x_i)$. Let $x(t) = tx_i + (1-t)\overline{x_i}$. $x(t) \in X_i$ by convexity, and $p^* \cdot x(t) < p^* \cdot x_i^*$ when $t < 1$. By continuity, $u_i(x(t)) > u_i(x_i^*)$ for t close to 1. This contradicts the definition of quasi-equilibrium, because $x(t)$ costs strictly less than x_i^* and is preferred to it. ∎

Definition Suppose that for any attainable allocation, x^*, and any (non-empty) partition, I_1, I_2, of agents, there is an attainable allocation, x, that makes some $i \in I_1$ better off and no $i \in I_1$ worse off. Then, we say that the economy is *resource related* or is *connected*. (That is, you can always transfer resources from I_2 to I_1 to make someone in I_1 better off.)

Proposition 2.10 McKenzie. *Suppose an economy satisfies resource relatedness and $\sum_{i=1}^{I} e_i \in Interior(\sum_{i=1}^{I} X_i)$. Then, every quasi-equilibrium is an equilibrium.*

Proof Since the total endowment lies in the interior and since prices must be non-zero, by attainability, there exists i such that $p^* x_i^* > \inf(p^* \cdot x_i)$. Let $I_1 = \{i : p^* \cdot x_i^* > \inf p^* \cdot x_i\}$, and let I_2 be all the other individuals. By resource relatedness, we can make the individuals in I_1 better off. Since they were already maximizing utility, this can only occur by increasing the value, $p^* \cdot x_i^*$, of an individual's consumption. This can only be done by taking value from I_2, which means that they must have had $p^* \cdot x_j^* > \inf(p^* \cdot x_j)$. This contradicts the definition of I_2, so I_2 must be empty. Then, we must have $p^* x_i^* > \inf(p^* x_i)$ for all $i \in I$; this reduces us to the previous case in which every quasi-equilibrium was also an equilibrium. ∎.

2.1.1 The Edgeworth Box

The Edgeworth box shows an exchange economy with two goods and two agents. The height and width of the box are determined by the total amount of the two goods in the economy, and the agents have their origins in opposite corners of the box. Prices are given by straight lines; if they go through the initial allocation, then they split the box into the two budget constraints. We want to find prices such that the indifference curves are tangent to the price line at the same point.

2.1.2 Quasi-Linear Preferences

With quasi-linear preferences in an exchange economy, the commodity space is R^{l+1}, where the $l + 1^{st}$ commodity is distinguished and called money (or the *numeraire*). The consumption set i s $R_+^l \times R$, so that money can be consumed in negative amounts, but everything else can only be consumed in non-negative quantities. Utility is quasi-linear in money, so that total utility is given by $u(x, m) = u(x) + m$, where $x \in R^l$. We assume that $u(0) = 0$. We normalize prices by assuming that $p_{l+1} = 1$.

The consumer's problem is to maximize $u(c) + m$ subject to $pc + m = w$, where w is the initial wealth. This gives the first order condition $u'(c) = p$. For a single commodity, we may graph the inverse demand function. Note that the *consumer surplus* is the area above the price and below the demand function, since it is $\int_0^{c^*} u'(c) - p = u(c^*) - pc^*$. This measures the gain from trade, since total utility moves from w to $u(c^*) - pc^* + w$. All of these results depend on the fact that utility is quasi-linear in money.

In this form of the pure exchange economy, initial endowments are given by $(e_i, \overline{m_i})$.

The total gains from trade are given by:

$$\sum_{i=1}^{I}(u_i(x_I) + m_i - u_i(e_i) - \overline{m_i}) = \sum_{i=1}^{I}(u_i(x_i) - u_i(e_i))$$

because the money cancels out. Thus, trading only money will not increase total welfare.

Theorem 2.11 *With quasi-linear preferences, an attainable allocation,* (x^*, m^*)*, is Pareto efficient if and only if it maximizes surplus.*

Proof ($\Rightarrow$) Suppose (x^*, m^*) is attainable and does not maximize surplus. Then, there is some (x^{**}, m^{**}) such that

$$\begin{aligned}\sum_{i=1}^{I}(u_i(x_i^{**}) - u_i(e_i)) &> \sum_{i=1}^{I}(u_i(x_i^{*}) - u_i(e_i)) \\ \sum_{i=1}^{I}u_i(x_i^{**}) &= \sum_{i=1}^{I}u_i(x_i^{*}) + a\end{aligned}$$

with $a > 0$. We redistribute the money:

$$\begin{aligned}u_i(x_i^{**}) + \tilde{m}_i &= u_i(x_i^*) + m_i^* + \frac{a}{I} \\ \tilde{m}_i &= u_i(x_i^*) - u_i(x_i^{**}) + m_i^* + \frac{a}{I}\end{aligned}$$

Then, $\sum_{i=1}^{I} \tilde{m}_i = -a + \sum_{i=1}^{I} m_i^* + a$, which is still feasible. Thus, $\tilde{m}$ gives a way to redistribute the money and increase total utility.

($\Leftarrow$) Suppose (x^{**}, m^{**}) maximizes surplus. Then, $u_i(x_i^{**}) + m_i^{**} \geq u_i(x_i^*) + m_i^*$ and this inequality is strict for at least one i. Then, $\sum_{i=1}^{I} u_i(x_i^{**}) > \sum_{i=1}^{I} u_i(x_i^*)$. ∎

Thus, it is sufficient to maximize $\sum_{i=1}^{I} u_i(x_i)$ subject to the constraint that $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i$, and $x_i \in R_+^l$.

Because we may redistribute the money, there are infinitely many Pareto efficient allocations. This means that the efficient allocation of goods is independent of welfare, because we may redistribute the m_i (subject to feasibility) without affecting the efficiency of the allocation. This is called *transferable utility.*

For quasi-linear utility, the utility possibility set is of the form $\mathcal{U} = \{u : \sum_{i=1}^I u_i \leq \max \sum_{i=1}^I u_i(x_i) + \sum_{i=1}^I \overline{m}_i\}$. The utility possibilities frontier is a plane, parameterized by m.

To find an efficient allocation for the first l goods, we use the Kuhn-Tucker theorem to minimize $-\sum_{i=1}^I u_i(x_i)$ subject to the constraints, $g_h(x) = \sum_{i=1}^I (x_{ih} - e_{ih}) \leq 0$ and $g_{ih}(x) = -x_{ih} \leq 0$ for $i = 1, ..., I$ and $h = 1, ..., l$. The necessary condition is that, for each i:

$$\begin{aligned} 0 &= -\sum_{j=1}^{I} \frac{\partial u_j}{\partial x_{ih}}(x_j) + \sum_{h=1}^{l} p_h \frac{\partial g_h}{\partial x_{ih}}(x_i h) + \sum_{j=1}^{J} \sum_{h=1}^{l} \mu_{jh} \frac{\partial g_{jh}}{\partial x_{ih}}(x_i) \\ &= \frac{\partial u_i}{\partial x_{ih}}(x_i) + \sum_{h=1}^{l} p_h - \mu_{ih} \end{aligned}$$

(since the utility of one individual does not affect the utility of another, by assumption). The complementary slackness conditions are $p_h \cdot \sum_{i=1}^I (x_{ih} - e_{ih}) = 0$ and $\mu_{ih} x_{ih} = 0$. If $x_{ih} > 0$, then $p_h = \frac{\partial u_i}{\partial x_{ih}}(x)$ for all i, so that all consumers have the same marginal utility of the h^{th} good, if they consume any of it. If $x_{ih} = 0$, then $\mu_{ih} \geq 0$, and $\frac{\partial u_i}{\partial x_{ih}}(x_i) \leq p_h$. Thus, $\frac{\partial u_i}{\partial x_{ih}}(x_i) \leq p_h$ for all i, h, with equality if $x_{ih} > 0$.

If the set of attainable allocations is convex and the utility functions are concave, then the necessary conditions are sufficient for an allocation to be efficient as well.

Note that, if the p_h were treated as fixed, these would be the same first order conditions as those for an individual maximizing $U(x_i) - p \cdot x_i$, or equivalently $u(x_i) + m_i$ subject to $p \cdot x_i + m_i = p \cdot e_i + \overline{m}_i$. The Kuhn-Tucker conditions for $U(x_i) - p \cdot x_i$ are:

$$\begin{aligned} -\nabla U_i(x_i) + p - \mu = & \; 0 \\ \mu_h x_{ih} = 0 & \end{aligned}$$

This is equivalent to $\sum_{h=1}^l (\frac{\partial u_i}{\partial x_{ih}}(x_i) - p_h) x_{ih} = 0$. (This is a special case of decentralization.)

2.2 Economies with Production

Definition A *production function*, $y = f(-l)$, describes the amount of an output that can be produced by l units of input.

By convention, inputs are described by negative numbers and outputs are described by positive numbers (this makes adding the result to the overall economy make sense, and makes profit calculation easier). In general, a good may be either an input or an output (and the choice may depend on relative prices).

Definition A *production set* or *technology*, $Y \subset R^l$, consists of points, y, that are feasible production vectors, where the negative numbers correspond to inputs and the positive numbers correspond to outputs.

Definition Let $p \in R^l_+$ be a price vector. Then, the *profit* at $y \in Y$ is given by $p \cdot y = \sum_{h=1}^{l} p_h y_h$. (This is the revenues generated by selling the outputs less the costs of buying the inputs.)

We maximize profits at the point where the isoprofit line (where $p \cdot y$ is constant) is tangent to the technology.

Some common assumptions about the production technology include:

- Y is non-empty.
- Y is closed, so that the limit of any sequence of feasible points is also feasible.
- $0 \in Y$, so that *inactivity* is an option. (This may not be true if there are fixed costs.)
- $Y \cap R^l_+ \subseteq \{0\}$, so that one cannot make output without any input. (This might hold for individual firms, but not for the economy as a whole.)
- *Free disposal*: $Y - R^l_+ \subset Y$. That is, if $y \in Y$ and $z \in R^l_+$, then $y - z \in Y$.
- *Non-increasing returns to scale*: If $\alpha \in (0, 1)$ and $y \in Y$, then $\alpha y \in Y$.
- *Constant returns to scale*: If $y \in Y$, $\alpha \geq 0$, then $\alpha y \in Y$. This implies that the boundaries of Y are straight lines, though they might change slope at the origin.
- *Additivity*: If $y, y' \in Y$, then $y + y' \in Y$.
- *Convexity*: If $\alpha \in (0, 1)$ and $y, y' \in Y$, then $\alpha y + (1 - \alpha)y' \in Y$.

(In general, we assume only that Y is non-empty, closed, convex, and contains 0; the other assumptions are less common.)

Proposition 2.12 *Non-increasing returns to scale and additivity imply convexity.*

Proof Let $y, y' \in Y$. By non-increasing returns to scale, $\alpha y, (1 - \alpha)y' \in Y$. By additivity, their sum is in Y, and Y is convex. ∎

Proposition 2.13 *Convexity and constant returns to scale imply additivity.*

Definition Let Y be a closed, non-empty, convex production set. A point, $y^* \in Y$ *maximizes profits* at a price vector, p, if $p \cdot y^* \geq p \cdot y$ for all $y \in Y$. (This is where the supporting hyperplane is normal to p.)

Definition A production plan y^* is *efficient* if it is feasible (that is, $y^* \in Y$) and if there does not exist any $y \in Y$ such that $y > y^*$ (that is, $y_h \geq y_h^*$ for all $h = 1, ..., l$ and $y_h > y_h^*$ for at least one h). That is, at an efficient point, one cannot decrease the input or increase the output without affecting the amount of other components used.

Proposition 2.14 *If p^* is strictly positive and y^* maximizes profits at p^*, then y^* is efficient.*

Proof Suppose y^* is not efficient. Then, there exists $y \in Y$ with $y > y^*$. Since p^* is strictly positive, $p^* \cdot y > p^* \cdot y^*$. This is a contradiction. ∎

Proposition 2.15 *If Y is convex and $y^* \in Y$ is efficient, then there exists $p^* \neq 0$ such that y^* is profit maximizing at p^*.*

Proof Suppose y^* is efficient. Then, $y^* \in Boundary(\overline{Y})$. By the supporting hyperplanes theorem, there is a supporting hyperplane such that $p^* \cdot y^* \geq p^* \cdot y$ for all $y \in Y$. ∎

Definition Suppose we have firms $j = 1, ..., n$, with production technology Y_j respectively. The *aggregate production set*, Y, is given by $Y = \sum_{j=1}^{n} Y_j$. That is, $y \in Y$ if $y = \sum_{j=1}^{n} y_j$, with each $y_j \in Y_j$.

The fact that the firm technologies are convex does not imply that the aggregate production set is convex. This definition also assumes that the choice of $y_j \in Y_j$ does not affect $Y_k, k \neq j$.

Proposition 2.16 *$y = \sum_{j=1}^{n} y_j$ is profit-maximizing with price vector p if and only if each y_j is profit maximizing in Y_j for price vector p.*

Proof ($\Rightarrow$) Suppose y is profit-maximizing but some y_j is not. Then, there exists some $y_j' \in Y_j$ such that $p^* \cdot y_j' > p^* \cdot y_j$. Define $y' = y_j' + \sum_{k \neq j} y_k$. Then, $p^* \cdot y' > p^* \cdot y$, which is a contradiction.

($\Leftarrow$) Suppose that each y_j is maximizing profit, but there is some $y' \in Y$ with $p^* \cdot y' > p^* \cdot y$. Then, $y' = \sum_{j=1}^{n} y_j'$ with each $y_j' \in Y_j$. We must have $p^* \cdot y_j' > p^* \cdot y_j$ for some j, which is a contradiction. ∎

Proposition 2.17 *$y \in Y$ is efficient if and only if $y_j \in Y_j$ is efficient for each $j = 1, .., n$.*

Proposition 2.18 *If p is strictly positive and y is profit-maximizing with $y = \sum_{j=1}^{n} y_j$, then for all j, y_j is profit-maximizing and efficient. If Y is convex and $y \in Y$ is efficient, there exists some p^* such that y is profit-maximizing and each y_j^* is efficient and profit-maximizing.*

The individual Y_j need not be convex for this result to hold. These propositions reduce n profit maximization problems to a single problem. This means that a single price vector decentralizes an efficient outcome for the whole economy.

Definition An *economy with production* consists of:

- l commodities,
- n firms, each with production set Y_j, and
- m consumers, each with a consumption set, X_i, a utility function, $u_i : X_i \to R$, an endowment, $e_i \in X_i$, and a portfolio of shares, $\theta_i \in R^n_+$, such that $\sum_{i=1}^m \theta_i = \begin{pmatrix} 1 \\ ... \\ 1 \end{pmatrix}$.

Definition An *allocation* is a pair (x, y), where $x = (x_1, ..., x_m)$ with each $x_i \in X_i$ and $y = (y_1, ..., y_n)$ with each $y_j \in Y_j$. An allocation is *attainable* if $\sum_{i=1}^m x_i = \sum_{i=1}^m e_i + \sum_{j=1}^n y_j$.

Definition A *Walrasian equilibrium* is an attainable allocation, (x^*, y^*) and a price vector, p^*, such that:

- firms are maximizing profits at y^*; that is, $\pi_j = p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$, $j = 1, .., n$, and
- consumers are maximizing utility; that is, $u(x_i^*) \geq u(x_i)$ for all $x_i \in X_i$ with $p^* \cdot x_i^* \leq p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} \pi_j$, where $\pi_j = p^* \cdot y_j$.

Theorem 2.19 *If preferences are locally non-satiable, then any equilibrium is efficient.*

Proof *(Sketch.)* Suppose there is some x_i such that $u(x_i) > u(x_i^*)$. Then, by a revealed preferences argument, $p^* x_i > p^* e_i + \sum_{j=1}^n \theta_i p^* y_j^*$. Summing over all consumers, we find that $\sum_{i=1}^m p^* x_i > \sum_{i=1}^m p^* e_i + \sum_{j=1}^n \sum_{i=1}^m \theta_i p^* y_j$. This contradicts feasibility. Furthermore, adding up shows that $\sum p^* \cdot (x_i - e_i) > \sum p^* \cdot y_j$, which means that firms must not be profit-maximizing in this case. ∎

Theorem 2.20 *Under some conditions, any equilibrium can be decentralized.*

Proof *Sketch.* In this proof, instead of showing that $0 \in Z$ for a contradiction, we show that $Y \cap Z = \emptyset$. ∎

Proposition 2.21 *At equilibrium with constant returns to scale, the firm's profit is 0.*

2.3 Two by two production economies

Suppose we have two goods, $h = 1, 2$, and two factors of production, capital (K) and labor (L). We assume that consumers get no utility from capital or labor, so that they are provided inelastically and exogenously. We also assume that there is no initial endowment of the two final goods.

We assume that the technology, $Y_h = F_h(K_h, L_h)$ has constant returns to scale. Then, if $y_h = \frac{Y_h}{L_h}$ and $k_h = \frac{K_h}{L_h}$, we have $y_h = \frac{1}{L_h}F_h(K_h, L_h) = F_h(k_h, 1) = f_h(k_h)$. Because of constant returns to scale, the marginal rate of substitution depends only on the capital-labor ratio, not on the level of the two factors (the slope of the isoquant is constant along the rays). Thus, we may determine the optimal capital-labor ratio based only on prices, not on the scale.

We assume that the prices can be taken as given, and that any output can be traded at those prices. Therefore, we want to maximize the value of production (which finds the efficient allocation as well), $p_1F_1(K_1, L_1) + p_2F_2(K_2, L_2)$, subject to $K_1 + K_2 \leq K$ and $L_1 + L_2 \leq L$. We normalize $L = 1$, set $\lambda = \frac{L_1}{L}$, and work in per capita terms. Then, we maximize $p_1\lambda f_1(k_1) + p_2(1-\lambda)f_2(k_2)$ subject to $\lambda k_1 + (1-\lambda)k_2 \leq k$. We assume that f_h is continuously differentiable, increasing, and strictly concave, with $f_h(0) = 0$ and $f_h'(0) = \infty$ (Inada conditions).

In this model, if we have an interior solution, then the first order conditions (with respect to k_1, k_2, λ respectively) are:

$$\begin{aligned} p_1 f_1'(k_1) &= \mu \\ p_2 f_2'(k_2) &= \mu \\ p_1 f_1(k_1) - p_2 f_2(k_2) &= \mu(k_1 - k_2) \end{aligned}$$

This gives us two equations for the two unknowns, k_1^*, k_2^*; the solution that we find is feasible if k lies between the two (that is, if $0 \leq \lambda^* \leq 1$). This shows that the *marginal revenue products of capital*, $p_h f_h'(k_h)$ are equal across the two industries and that the difference in the industries' value of output per worker, $p_1 f_1(k_1) - p_2 f_2(k_2)$ is proportional to the difference in capital per worker. We may also substitute for μ to find:

$$p_1(f_1(k_1) - f_1'(k_1)k_1) = p_2(f_2(k_2) - f_2'(k_2)k_2)$$

The *marginal revenue product of labor*, $p_h(f_h(k_h) - f_h'(k_h)k_h) = p_h\omega_h(k_h)$, must be equal across the two industries. Dividing the two equations shows that the *marginal rates of technical substitution* are equal:

$$\frac{f_1'(k_1)}{f_1(k_1) - f_1'(k_1)k_1} \frac{f_2'(k_2)}{f_2(k_2) - f_2'(k_2)k_2}$$

(DO THE MARGINAL RATES OF TRANSFORMATION ALSO HAVE TO BE EQUAL?)

Definition If $k_1 > k_2$ whenever the marginal rates of transformation are equal, then good 1 is *capital intensive* and good 2 is *labor intensive.* If the capital-labor ratios switch, this is called *factor intensity reversal.*

If one efficient interior point lies on the diagonal and there are constant returns to scale, then the entire diagonal is efficient by constant returns to scale. Otherwise, all the efficient points must lie on the same side of the diagonal, which determines which goods are capital and labor intensive. To make more of the capital-intensive good, the capital-labor ratios of both goods must fall. Note that $\omega_h'(k_h) = -f_h''(k_h)k_h > 0$ (since f is concave), so that when k_h increases, $\frac{f_h'(k_h)}{\omega_h(k_h)}$ decreases, and k_{3-h} must increase as well to preserve the equality. Thus, k_1, k_2 move together (though K_1, K_2 do not always move together).

Definition Assume that p_1, p_2 are given and we have an inelastic supply of labor and capital. An *allocation* in this economy consists of a pair $(k_1^*, l_1^*), (k_2^*, l_2^*)$. An allocation is *attainable* of $l_1^* + l_2^* = 1$ and $k_1^* l_1^* + k_2^* l_2^* = k$. An *equilibrium* consists of an attainable allocation and factor prices, (r^*, w^*), such that for $h = 1, 2$, (k_h^*, l_h^*) solves the producer's problem of maximizing $p_h f_h(k_h) l_h - r^* k_h l_h - w^* l_h$ subject to $k_h \geq 0$ and $l_h \geq 0$.

The first order conditions for producers are:

$$\begin{aligned} r^* &= p_h f'(k_h) \\ w^* &= p_h f_h(k_h) - r^* k_h = p_h(f_h(k_h) - f'(k_h)k_h) \end{aligned}$$

As before, we may find an efficient allocation as the central planner and then choose r^*, w^* to decentralize it.

In this model, total household income is $r^*K + w^*L$.

Applications to trade theory

Theorem 2.22 Rybcszynzki *If capital increases, then production of the capital-intensive goods will increase and labor allocated to the labor intensive good will decrease.*

Proof Since the capital-labor ratios of the two industries are determined by the first order conditions, not by K, they will not change. Therefore, the labor allocation must adjust so that $k_1^*\lambda + (1-\lambda)k_2^* = k$. ∎

Theorem 2.23 Stolper and Samuelson. *If the price of the capital intensive good increases and the equilibrium is in the interior, then the price (which equals the marginal revenue product) of capital will increase and the wage will decrease.*

Proof Using differentials, we find that $\lambda^*(dk_1^* - dk_2^*) + (k_1^* - k_2^*)d\lambda^* = 0$. In this case, $k_1^* - k_2^* > 0$, since the first good is capital intensive, and $d\lambda^* > 0$, because demand for the first good has increased. This means that the second term must be negative to offset the positive term. Since $\lambda^* > 0$, we must have $dk_1^*, dk_2^* < 0$.

In terms of factor prices, $r^* = p_1 f'(k_1^*)$, so if p_1 increases and k_1^* decreases, then $f'(k_1^*)$ increases and r increases. Since $w^* = p_2(f_2(k_2^*) - f_2'(k_2^*)k_2^*)$ and p_2 is unchanged while the second term decreases, wages must decrease. ∎

Theorem 2.24 Factor Price Equalization Theorem. *If output prices are equal in two countries, that is, $(p_1, p_2) = (p_1^*, p_2^*)$, then capital-labor ratios and therefore wages and rental prices will be equal in the two economies (even though the factors are not mobile).*

Theorem 2.25 Hecksher-Olin. *Each country exports the good that uses intensively the factor the country has in abundance.*

2.4 Computing Equilibria

With identical, homothetic preferences (or production), we way aggregate up to a representative consumer (producer), because increases in income shift the budget line but keep it parallel and demand for each good increases proportionately. That is, the individual's demand curve is $f_i(p, m_i) = m_i f_i(p, 1)$, so that the aggregate demand curve is $\sum_{i=1}^{I} f_i(p, m_i) = f(p, 1) \sum_{i=1}^{I} m_i$.

Though we usually compute with one agent (or two), we really believe that the single representative consumer represents a continuum of consumers. Then, changing the behavior of one will have a negligible effect on everyone else. We also assume that it is possible to choose everything simultaneously, even though the decision of one agent may affect the constraints of another.

To show that a planner's solution is an equilibrium, show that it is attainable, and then find prices to decentralize the allocation, using the firm's first order conditions to determine r and w and the consumer's first order conditions and the budget constraint to determine the prices of the outputs. (The budget constraint must be exactly satisfied.)

One way to compute an equilibrium (given initial endowments in an exchange economy):

1. For any (p_1, p_2), compute the quantity demanded by each consumer.
2. Find (p_1, p_2) such that the total demanded is feasible.

Once a candidate equilibrium is found, we must check that it satisfies:

- budget constraints,
- feasibility, and
- the consumer's first order conditions.

Don't forget to check the boundaries (that is, 0 prices and quantities) when looking for equilibria. Note that if preferences are increasing in good 1, then we cannot have $p_1 = 0$ because the preferred allocation would include an infinite amount of good 1, which is not feasible. Once the boundaries have been dealt with, the first order conditions are necessary and sufficient.

Computing a competitive equilibrium differs from decentralization because people's incomes (endowments) are fixed, and there are no lump sum transfers. This means that we are limited to budget sets that pass through the initial endowment.

A competitive equilibrium is unique with a single representative agent with strictly concave preferences. However, adding taxes may lead to multiple equilibria.

2.5 Existence and Uniqueness of Equilibria

Definition Let a set of prices, p, be given. An individual's *excess demand* at this set of prices is the difference between the quantity that the consumer wants to consume at these prices and the consumer's endowment, $z_i(p) = x_i(p) - e_i$. The *aggregate excess demand* is $z(p) = \sum_{i=1}^{I} z_i(p)$.

2.5.1 Existence

In a pure exchange economy, at equilibrium, $z(p) = 0$. In a pure exchange economy with exactly two goods, then we may parameterize $z(p)$ by p_1, setting $p_2 = 1 - p_1$, and consider only the first component (since, by Walras's law that $p \cdot z_i(p) = p \cdot z(p) = 0$, excess demand for one good is 0 if and only if excess demand for the other good is zero). If excess demand for the first good is positive when $p_1 = 0$ and negative when $p_2 = 0$ and excess demand is continuous, then there must be at least one equilibrium, when the excess demand function crosses 0.

Theorem 2.26 *Suppose $\mathcal{E} = \{(X_i, e_i, u_i)\}_{i=1}^{M}$ is an exchange economy with $X_i = R^l_+$, $e_i >> 0$, $e_i : R^l_+ \to R$ increasing, continuous, and strictly concave. Define $P = \{p \in R^l_+ : p >> 0, p_l = 1\}$ (the set of strictly positive prices with the l^{th} good as the numeraire) and $\overline{P} = R^l_+$. Define the budget set by $B_i(p, w_i) = \{x \in R^l_+ : p \cdot x_i \leq w_i\}$. Define the demand function by $\xi_i(p) = argmax\{u_i(x_i) : x_i \in B_i(p, p \cdot e_i)\}$. Then, $B(p, w_i)$ is compact, and $\xi_i(p)$ is a well-defined, continuous function.*

Proof Choose $p^q \in P$, such that $p^q \to p^0 \in P$. Since $\{\xi_i(p^q)\}$ is bounded, it has a convergent subsequence. If $\xi_i(p^q) \to x$, then $x \in R^l_+$ and $x \in B_i(p^0, p^0 \cdot e_i)$. Suppose $x \neq \xi_i(p^0)$. Then, $u_i(x) < u_i(\xi_i(p^0))$. Since $p^0 \cdot e_i > 0$, either $\xi_i(p^q) \in Interior(B_i(p^0, p^0 \cdot e_i))$ or there exists $y \in Interior(B_i(p^0, p^0 \cdot e_i))$ with $u(x) < u(y) < u_i(\xi_i(p^0))$. For large enough q, $y \in B_i(p^q, p^q \cdot e_i)$, and $u(y) > u(\xi_i(p^q))$. This contradicts the optimality of $\xi_i(p^q)$. ∎

Proposition 2.27 *Let $z_i(p) = \xi_i(p) - e_i$ be the individuals excess demand function. Suppose $p^q \in P$ and $p^q \to p^0 \in \overline{P} - P$. Then, $\|z_i(p)\| \to \infty$.*

Proposition 2.28 *Let $z(p) = \sum_{i=1}^{I} z_i(p)$ be the aggregate excess demand function. Then, $z : P \to R^l$ is a well-defined, continuous function. For any $p^q \in P$ such that $p^q \to p^0 \in \overline{P} - P$, $\|z(p^q)\| \to \infty$.*

Consider the excess demand function, $Z(p, e_1, ..., e_m)$. If ∇Z has rank n (which is the number of goods) then for almost every set of initial endowments, by the Transversality Theorem, $\nabla_p Z(p, e)$ has full rank. This means that equilibria are locally unique.

Consider the simplex of normalized prices, $\{p \in R_l^+ : \sum_{h=1}^{l} p_h = 1\}$. Let P be the interior of this set (so that all prices are non-zero). Let $f : P \to R^l$ be the excess demand function. Suppose f is continuous and satisfies Walras's law and that $\|f(p)\| \to \infty$ as p goes to the boundary of P (where some prices are 0). Then, there is some $p^* \in P$ such that $f(p^*) = 0$; this is an equilibrium.

Let $P^q \subset P$ be compact. Consider the mapping $z \to \mu(z) = \{p \in P^q : p \cdot z \geq p' \cdot z \text{ for all } p' \in P^q\}$. This mapping satisfied Kakutani's Theorem, and therefore there is a fixed point. As $P^q \to P$, this fixed point must converge to the fixed for the the entire set, which no longer needs to be compact.

Equilibria may fail to exist in some cases, including:

- *Non-convexity of preferences*: Suppose the initial endowment is Pareto efficient. Then, this is the only possibly equilibrium. If there are no prices that decentralize this particular equilibrium because of non-convexity, there can be no equilibrium.
- *Satiation*: If the sum of all the bliss points is less than the total endowment, only negative prices would decentralize the equilibrium.
- *Resource relatedness*: Suppose $I = 2$, $u_1(x, y) = y, u_2(x, y) = x, e_1 = (1, 0), e_2 = (1, 0)$. Resource relatedness fails because there is no way to transfer goods from 2 to 1 that will make him better off. We must have $p_1 > 0$, or else the second individual would demand an infinite amount. However, the first individual would then demand $\frac{p_1}{p_2}$ of good 2, which violates feasibility.

2.5.2 Uniqueness

Uniqueness is helpful for prediction and for comparative statics (otherwise, a policy change could lead to an entirely new equilibrium).

Proposition 2.29 *Suppose production has constant returns to scale (so that it is enough to consider $Y = \sum_{j=1}^{n} Y_j$, which is a convex set containing 0; more generally, we may assume that Y is convex). Suppose we have an equilibrium price vector, $p^* \in P$, such that markets clear (that is, $z(p^*) \in Y$) and firms have non-positive profits (that is, for all $y \in Y$, we have $p^* \cdot y \leq 0$). If there is a representative agent (because there is only one agent or because agents have identical homothetic preferences), then there is a unique equilibrium.*

Proof If this is a pure exchange economy, then we must have $x_1 = e_1$ since there is only one agent. If u_1 is differentiable, then this allocation uniquely determines $p^* = \alpha \nabla u_i(e_i)$ (up to a scalar multiple). If there is production, then we maximize $u_1(x_1)$ subject to $x_1 - e_1 \in Y$. If Y is convex and u_1 is strictly concave, then $x_1$is unique. If u_1 is differentiable, then the supporting prices are unique as well. ∎

Definition Suppose $p, p' \in P$ and $p \neq \alpha p'$ for any α. Suppose $p_h = p'_h$ for all $h \neq k$ and $p_k > p'_k$. The *gross substitutes property* holds if $z_h(p) > z_h(p')$ for all $h \neq k$ (and, therefore, $z_k(p) < z_k(p')$ by Walras's Law).

Theorem 2.30 *If the gross substitutes property holds, then equilibrium prices are unique.*

Proof (Pure exchange case.) Suppose $p \neq p'$ and $z(p) = z(p') = 0$. Without loss of generality, we may relabel the goods and rescale so that $p_l = p'_l$ and $p_h \geq p'_h$ for all $h = 1, ..., l$. Suppose we reduce p_1 to p'_1. Then, demand for goods $2, ..., l$ will decrease by the gross substitutes property. Suppose we reduce p_h to p'_h, then demand for all goods but h will decrease. If we do this for $h = 1, ..., l-1$, then demand for good l must decrease in every step, and $z_l(p') < z_l(p) = 0$. This contradicts the assumption that p' is an equilibrium. Thus, equilibrium prices are unique. ∎

Definition Suppose we have prices p, p' and $z(p) \neq z(p')$. If the *weak axiom of revealed preferences (WARP)* holds, then $p \cdot z(p') \leq 0$ implies that $p' \cdot z(p) > 0$. That is, if $z(p')$ satisfies the budget constraints at p (but is not chosen), then $z(p)$ must be too expensive at prices p' (or it would be chosen).

Theorem 2.31 *If the weak axiom holds, then the set of equilibrium prices is convex.*

Proof Suppose $p \neq p'$ are equilibria. Let $p'' = \alpha p + (1-\alpha)p'$, $\alpha \in (0,1)$. By Walras's Law:

$$0 = p'' \cdot z(p'') = \alpha p \cdot z(p'') + (1-\alpha)p' \cdot z(p'')$$

At least one of these terms is non-positive. Without loss of generality, assume that $p \cdot z(p'') \leq 0$.

Suppose $z(p) \neq z(p'')$. Then, by the weak axiom, $p'' \cdot z(p) > 0$. Then, we find that:

$$0 < (\alpha p + (1-\alpha)p') \cdot z(p) = \alpha p \cdot z(p) + (1-\alpha)p' \cdot z(p)$$

Since the first term is 0 by Walras' law, we must have $0 < p' \cdot z(p)$. Since this is an equilibrium, $z(p) \in Y$. By the definition of an equilibrium, $p'y \leq 0$ for all $y \in Y$, which would mean that $p'z(p) \leq 0$, which is a contradiction.

Thus, $z(p'') = z(p) \in Y$, and $z(p'')$ must be an equilibrium as well. (This also means that all equilibria must have the same production.) ∎

3 Uncertainty and Incomplete Markets

Suppose we have $h = 1, .., l$ physical goods and $s = 1, ..., S$ possible states of nature. Then, there are $l \cdot S$ contingent commodities, (h, s). In the most general case, we allow people to have arbitrary preferences over the lS goods; these preferences may include subjective probabilities and risk preferences.

Definition Suppose we have an exchange economy, $\mathcal{E}$, with

- Commodity space, R^{lS}_+
- $i = 1, ..., m$ individuals, each with consumption set R^{lS}_+, utility function, $u_i : R^{lS}_+ \to R$, and endowment $e_i \in R^{lS}_+$

A *Walrasian* or *Arrow-Debreu equilibrium* consists of an attainable allocation, $x^* = (x_1^*, ..., x_m^*)$, and a price vector, p^*, such that each x_i^* maximizes $u_i(x_i)$ subject to $p^* \cdot x_i \le p^* \cdot e_i$ and $x_i \in R_+^{lS}$.

In a Walrasian equilibrium, all lS goods are traded before the state is realized. After the state is realized, the state contingent goods are delivered (but no further trading need occur). This sort of equilibrium is identical to the previous sections. Note that, in any given state, the individual demands $x_{is}^* - e_{is}$. The trades of goods contingent on one state pay for goods contingent on other states. That is, $\sum_{s=1}^S p_s^* \cdot x_{is}^* \le \sum_{s=1}^S p_s^* \cdot e_{is}$.

By attainability, $\sum_{i=1}^m x_{is}^* = \sum_{i=1}^m e_{is}^*$ for all s, and the market must clear in each state.

3.1 Equilibrium with Arrow Securities

Definition An *Arrow security* is a promise to deliver one unit of account in state s and nothing in any state $s' \neq s$.

In a market with Arrow securities, we have the following process:

- **Time 0**: At this time, there are S markets for S securities. Let $q \in R_+^S$ be the vector of securities prices. Each individual creates a portfolio of securities, $z_i = (z_{i1}, ..., z_{iS})$. The budget constraint at time 0 is $q \cdot z_i \le 0$. The market clearing condition is $\sum_{i=1}^I z_i = 0$.
- **Time 1**: For the realized state, s, we have *spot prices*, $p_s = (p_{s1}, ..., p_{sl}) \in R_+^l$. Each individual has endowment $e_{is} = (e_{is1}, ..., e_{isl})$ and budget constraint $p_s \cdot x_{is} \le p_s \cdot e_{is} + z_{is}$. Individuals demand $x_{is} = (x_{is1}, ..., x_{isl})$.

Definition An *attainable allocation* is (x, z) such that $x_i \in R_+^{lS}$, $z_i \in R^S$,$\sum_{i=1}^I z_i = 0$ and $\sum_{i=1}^I x_i = \sum_{i=1}^I e_i$.

The securities allow people to redistribute income across states. This assumes that people (correctly) anticipate prices across all states in time 1 when they make their decisions about securities.

We may think of the x_i as consumption plans; if state s occurs, then only x_{is} is realized.

Definition An *equilibrium with Arrow securities* consists of an attainable allocation, (x^*, z^*), and a price system, (p^*, q^*), such that for any i, (x_i^*, z_i^*) maximizes $u_i(x_i)$ subject to the constraints:

$$\begin{aligned} x_i &\in R_+^{lS} \\ q^* \cdot z_i &\le 0 \\ x_{is}^* \cdot p_s^* &\le p_s^* \cdot e_{is}^* + z_{is} \end{aligned}$$

Note that $u_i(x_i)$ need not be separable across states, so considering the utility of a single state may not make sense. With Von-Neumann Morgenstern utilities, however, $u_i(x_i) = \sum_{s=1}^{S} \pi_{is} V_i(x_{is})$. This could lead to a two-step budgeting process, in which the consumer first computes their indirect utility, $v_i(p_s^*, p_s^* \cdot e_{is} + z_{is})$, for an arbitrary z_{is} and then maximizes the expected indirect utility, $\sum_{s=1}^{S} \pi_{is} v_i(p_s^*, p_s^* \cdot e_{is} + z_{is})$ subject to the constraint, $q^* \cdot z_i \leq 0$.

Theorem 3.1 *Suppose (x^*, p^*) is a Walrasian equilibrium. Then, there is an equilibrium with Arrow securities, (x^*, z^*, p^*, q^*), for some q^*, z^*.*

Proof Define $q^* = (1, ..., 1)$. Define $z_{is}^* = p_s^* \cdot (x_{is}^* - e_{is})$. We check that (x^*, z^*, p^*, q^*) satisfies the required properties:

- $q^* \cdot z_i^* = \sum_{s=1}^{S} p_s^* \cdot (x_{is}^* - e_{is}) = p^* \cdot x_i^* - p^* \cdot e_i^* = 0$, because this is a Walrasian equilibrium.
- $p_s^* \cdot x_{is}^* \leq p_s^* \cdot e_{is} + z_{is}^*$ by construction.
- Any point in the Arrow securities budget set belongs to the Walrasian budget set (since $q^* \cdot z \leq 0$ implies that $p^* \cdot x \leq p^* \cdot e_i$ for any (x, z)). Since x_i^* is a Walrasian equilibrium, it is preferred to all points in the Walrasian budget set, and therefore all points in the Arrow securities budget set.
- (x^*, z^*) is attainable, because $\sum_{i=1}^{I} z_{is}^* = \sum_{i=1}^{I} p_s^* \cdot (x_{is}^* - e_{is}) = 0$.

Thus, (x^*, z^*, p^*, q^*) is an equilibrium with Arrow securities. ▮

Theorem 3.2 *Suppose (x^*, z^*, p^*, q^*) is an equilibrium with Arrow securities. Then, there exists $\hat{p}^*$ such that $(x^*, \hat{p}^*)$ is a Walrasian equilibrium.*

Proof Define $\hat{p}_s^* = q_s^* p_s^*$ for $s = 1, .., S$. We check that this is a Walrasian equilibrium.

- By attainability, $z_{is}^* = p_s^* \cdot (x_{is}^* - e_{is}^*)$. Then,

$$0 \geq \sum_{s=1}^{S} q_s^* z_{is}^* = \sum_{s=1}^{S} q_s^* p_s^* \cdot (x_{is}^* - e_{is}^*) = \hat{p}^* \cdot (x_i^* - e_i)$$

  and the budget constraint is satisfied.
- Suppose x_i is in the Walrasian budget set. Then, $\hat{p}^* \cdot x_i \leq \hat{p}^* \cdot e_i$. Set $z_{is} = p_s^* \cdot x_{is} - p_s^* \cdot e_{is}$. Then, $\sum_{s=1}^{S} q_s^* z_{is} = \sum_{s=1}^{S} q_s^* p_s^* \cdot (x_{is} - e_{is}) \leq 0$, and (x, z) is in the Arrow securities budget set. Since (x^*, z^*) is the optimal point, we must have $u_i(x_i) \leq u_i(x_i^*)$.
- x^* is attainable.

Thus, $(x^*, \hat{p}^*)$ is a Walrasian equilibrium. ▮

In an equilibrium with Arrow securities, the price levels may differ across the states without changing the allocation. These differences affect q, so that z_{is} adjusts to change with the price levels. q defines "exchange rates" across the states.

In a Walrasian equilibrium or an equilibrium with Arrow securities with Von Neumann-Morgenstern utility, we must have either $p(s) \propto \pi_i(s)\nabla u_i(x_i(s))$ or $q_s p(s) \propto \pi_i(s)\nabla u_i(x_i(s))$. Thus, the probabilities of each state affect the prices across states.

3.2 Radner Equilibrium

Definition A *security* is a claim to a bundle of commodities which can be traded.

Definition Suppose there are n securities. A *portfolio*, $\theta \in R^n$, is a vector which describes how many of each security is held. (Some θ_j can be negative.)

Definition The commodity bundle associated with a portfolio, θ, is given by $x = \sum_{j=1}^{n} \theta_j y_j$. The space *spanned by the securities* is $X = \{x : x = \sum_{j=1}^{n} \theta_j y_j \text{ for some } \theta \in R^n\}$.

Definition A security is *redundant* if it can be written as a linear combination of the other securities.

In terms of markets, one could remove a redundant security and not change the outcome, because it could be synthesized by a portfolio.

Proposition 3.3 *There is a redundant security if and only if the dimension of X is less than n.*

Proof Possibly with relabeling, we may choose a basis of X, $\{y_1, ..., y_k\}$, $k < n$. Then, $y_{k+1}, ..., y_n$ are redundant, because they are linear combinations of the basis. ∎

Definition Suppose we have securities prices, $q = (q_1, ..., q_n)$. The *value of a portfolio*, θ, is given by $q \cdot \theta = \sum_{j=1}^{n} q_j \theta_j$. An *arbitrage* exists for a price vector q if there is some portfolio, θ, such that $q \cdot \theta > 0$ but $\sum_{j=1}^{n} \theta_j y_j = 0$. That is, this portfolio has a positive value and zero commodities, and therefore is pure profit.

In equilibrium, arbitrage should not exist. Otherwise, everyone would take advantage of it and have unbounded income. Even without equilibrium, one might assume that there is no arbitrage, which leads to *arbitrage pricing theory.*

Proposition 3.4 *There is no arbitrage if and only if there exists prices of the underlying commodities, $p \in R^l$, such that $q_j = p \cdot y_j$.*

Proposition 3.5 *We can price redundant securities from the basis.*

Proof Let $\{y_1, ..., y_k\}$ be a basis. For all $i > k$, we may write $y_i = \sum_{j=1}^{k} \alpha_{ij} y_j$. Since there is no arbitrage:

$$q_i = p \cdot y_i = \sum_{j=1}^{k} \alpha_{ij} p \cdot y_j = \sum_{j=1}^{k} q_j \alpha_{ij}$$

Suppose $q_i > \sum_{j=1}^{k} q_j \alpha_{ij}$. Then, the arbitrage portfolio would be $\theta = (\alpha_{i1}, ..., \alpha_{ik}, 0, ..., 1, ...0)$, where the 1 is in the i^{th} location. Then, $q \cdot \theta > 0$ and $\sum_{j=1}^{n} \theta_j y_j = 0$ and there is arbitrage. $\blacksquare$

In an exchange economy with assets, we have:

- I individuals with consumption sets, X_i, endowments, e_i, and utility functions, u_i,
- two periods, $t = 0, 1$, with states $s = 1, ..., S$ at time 1,
- l physical goods available in all times and states; this leads to $l(S+1) = L$ date and state dependent commodities,
- assets $k = 1, ..., K$, where $a_k(s)$ is the vector of goods that asset k promises in state s, and
- a market structure with:
  - spot markets for time 0 goods at time 0,
  - markets for assets at time 0, and
  - spot markets for time 1 (state s) goods at time 1.

Definition In this economy, an *allocation*, (x, z), consists of $x_i \in R^L_+$, $z_i \in R^K$. An allocation is *attainable* if $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i$ and $\sum_{i=1}^{I} z_i = 0$.

In this economy, assets can be used to transfer wealth across time and states; spot markets allocate that wealth to goods.

If $p(0)$ are spot prices at time 0, $p(s)$ are spot prices at time 1 in state s, and q are the prices of assets at time 0, then each household has the budget constraints:

$$\begin{aligned} p(0) \cdot (x_i(0) - e_i(0)) + q \cdot z_i &\leq 0 \\ p(s) \cdot (x_i(s) - e_i(s)) &\leq \sum_{k=1}^{K} z_{ik} p(s) \cdot a_k(s) \end{aligned}$$

Definition In a *Radner equilibrium*, there is an attainable allocation, (x^*, z^*) and a price system, (p^*, q^*), such that the allocation maximizes each individual's utility subject to their budget constraints.

Proposition 3.6 *Let $r_k(s) = p(s) \cdot a_k(s)$ (this is the* return *on the asset), and let R be the matrix of the $r_k(s)$. If the rank of R is S, then we have complete markets and the Radner equilibrium is equivalent to the Walrasian equilibrium.*

In general, one can find the Walrasian equilibrium and then check the rank of R. If it has full rank, then the Radner equilibrium has been found as well; otherwise, there are no theorems to simplify the calculation. Sometimes, no equilibrium exists.

As prices change, the rank of R changes discontinuously. If the columns are nearly collinear, then the positions needed to transfer the same wealth across states become more extreme. In the limit, the rank drops and some combinations of wealth are not attainable at all.

Definition An allocation, (x^*, z^*), is *constrained efficient* if it is not possible to make everyone better off by redistributing $(x_i(0), z_i)$ in any feasible way and then allowing people to trade at time 1.

This constrains the planner to work within the existing market structure, but allows the use of price knowledge at time 1 to make people better off.

Theorem 3.7 Geanakaplos-Polemarchakis. *Almost all incomplete equilibria are constrained inefficient.*

Also, adding new markets can even make people worse off.

4 Strategic Foundations of General Equilibrium

The strategic foundations of general equilibrium combine the description of an economy with an extensive form game in order to achieve the equilibrium outcome that matches the predictions of general equilibrium theory.

4.1 Equilibrium in Game Theory

Definition A *game* consists of:

- a set of players, $i = 1, ..., I$,
- a strategy set, X_i, for each player,
- strategy profiles, $X = X_1 \times ... \times X_I$, and
- a *payoff function*, $u : X \to R^I$, in which each strategy profile is mapped to a payoff for each player; we define $u_i(x_i, x_{-i})$ as the payoff when the i^{th} player chooses x_i and everyone else choose $x_{-i} \in R^{I-1}$.

Definition A *Nash equilibrium* is $x^* \in X$ such that $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i})$ for all $x_i \in X_i$ and $i = 1, ..., I$. That is, if everyone else's strategy is taken as given, then no individual can do better.

We generally assume that each X_i is non-empty, convex, and compact (and therefore X is as well). $u_i : X \to R$ is continuous, and $u_i(x_i, x_{-i})$ is quasi-concave in x_i for any fixed x_{-i}.

Definition The *best response correspondence* is $\phi_i(x_i, x_{-i}) = \{x_i' \in X_i | u_i(x_i', x_{-i}) \text{ is a maximum}\}$. (This correspondence does not depend on x_i, but we keep it in for later notational convenience.)

$\phi_i(x_i, x_{-i})$ is convex and non-empty for all x.

Proposition 4.1 *$\phi_i(x)$ has the closed graph property.*

Proof Suppose $x^q \to x^0$, $y^q \in \phi_i(x^q)$ and $y^q \to y^0$. Since X_i is compact, if $y^q \in X_i$ for all q, then $y^0 \in X_i$, and y^0 is feasible. Suppose y^0 is not a best response. Then, there is some $\overline{y}$ with $u_i(\overline{y}, x^0_{-i}) > u(y^0, x^0)$ (CHECK THIS NOTATION). Using limits, $u_i(y^q, x^q) \geq u_i(\overline{y}, x^q)$ (since each y^q is a best response), and then $u_i(y^0, x^0) \geq u_i(\overline{y}, x^0)$. This is a contradiction. ∎

Theorem 4.2 *There is always a Nash Equilibrium.*

Proof Let $\phi : x \to X$ be defined by $\phi(x) = (\phi_1(x), ..., \phi_I(x))$. Then, ϕ is convex, non-empty, and has the closed graph property. Therefore, we may apply the Kakutani Theorem to find that there exists $x^* \in \phi(x^*)$ which is a fixed point. At x^*, each player is choosing the best response; therefore, this is a Nash Equilibrium. ∎

This is also true in the case where X_i depends (continuously) on the choices of the other players.

In a more general setting, each player, i, may represent a continuum of identical individuals. This is useful for proving limit theorems.

Definition (Edgeworth.) The *contract curve* is the set of efficient points in which both individuals are at least as well off as they were with their initial endowments.

If individuals are individually rational, then the equilibrium should always be somewhere on the contract curve.

If there are multiple individuals of the same type in the economy, the contract curve will shrink. As the number goes to infinity, it will shrink to the competitive equilibrium.

Definition Suppose we have an exchange economy, $\mathcal{E} = \{(X_i, e_i, u_i)\}_{i=1}^I$. Let $S \subset \{1, ..., I\}$, $S \neq \emptyset$. We call S a *coalition.* Let x be an attainable allocation. S can *improve on* x if there exists an attainable allocation x' such that $u_i(x_i') \geq u_i(x_i)$ for all $i \in S$, with strict inequality for at least one i, and if $\sum_{i \in S} x_i' = \sum_{i \in S} e_i$ (so that this allocation does not depend on trading with the other individuals). The *core* of the exchange economy consists of the set of attainable allocations that cannot be improved on.

Definition The *Nash Program* tries to find non-cooperative games that match the outcome of cooperative games.

Definition Suppose there are two firms and that the price of their good in the economy depends on the sum of their output, $Q = q_1 + q_2$. A *Nash-Cournot equilibrium* is one in which each firm maximizes its revenue, holding the output of the other firm fixed.

The same sort of equilibrium could be found for n firms and n consumers, so that $P(\frac{Q}{n})$ is unchanged as n grows. Then, the representative firm maximizes $P(\frac{(n-1)q^*+q_i}{n})q_i$, which leads to the first order condition (for q^*) that $P'(\frac{nq^*}{n})\frac{q^*}{n} + P(\frac{nq^*}{n}) = 0$. Thus, the effect of each firm on the market decreases an s$n \to \infty$. In the limit, the first order condition is to produce where price equals marginal costs, which is the competitive equilibrium.

A model with bidding: Suppose we have an economy with I agents and $l+1$ commodities, where the $l + 1^{st}$ commodity is the medium of exchange. The method of exchange is:

1. All individuals offer their whole endowment of goods $1, ...l$ for exchange.
2. Individuals bid for the available goods $h = 1, ..., l$, where $b_{ih} \geq 0$ is a quantity of good $l+1$ offered in exchange for good h. Bids are constrained by e_{l+1} (a cash in advance constraint).
3. Each individual receives $x_{ih} = \frac{b_{ih}}{\sum_{j=1}^{I} b_{jh}} \sum_{j=1}^{I} e_{jh}$ at price $p_h = \frac{1}{\sum_{j=1}^{I} e_{jh}} \sum_{j=1}^{I} b_{jh}$.
4. Each individual received payment proportional to the bids that people pay. However, there is no way to exchange those payments for goods in future rounds of bidding.

Theorem 4.3 *Suppose each u_i is continuous, concave, and non-decreasing, and that for all $h = 1, .., l$, there are at least two agents with positive endowments of good $l+1$ whose utility is increasing in each h. Then, a Nash equilibrium exists.*