Microeconomics Summary

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Microeconomics is based on the decisions of individual agents. Each agent faces a *choice problem*, where a set of options is given and the individual chooses one. This assumes that individuals are aware of all their options, have a goal when they make the decision, and are *rational*, so that they make the best decision given the information and the goal.

1 Preferences

Preferences are defined over a set, X.

Model 1: Suppose we observe a function, $f(a,b) \in \{a > b, I, b > a\}$, where $a > b$ means that a is strictly preferred to b and I means indifferences. (Note that these are the only options; choices like "I don't know," "It depends", "Someone else decides", and "Strongly $a > b$" are not options.) We define the preferences based on this function to be *consistent* if:

- $f(a,b) = f(b,a)$
- *Transitivity* holds: If $f(a,b) = (a > b)$ and $f(b,c) = (b > c)$ then $f(a,c) = (a > c)$. If $f(a,b) = f(b,c) = I$, then $f(a,c) = I$.

Proposition 1.1 *If $f(a,b) = (a > b)$ and $f(b,c) = I$, then $f(a,c) = (a > c)$.*

Proof Suppose $f(a,c) = I$. Then, $f(a,c) = f(b,c) = I$ and we must have $f(a,b) = I$. Suppose $f(a,c) = (c > a)$. Then, $f(a,b) = (a > b)$ and we must have $f(b,c) = (c > b)$. Either possibility contradicts the assumptions of the proposition, so we must have $f(a,c) = (c > a)$. ∎

Note that consistency can be violated for psychological reasons (such as saying one is indifferent between pairs that are "about the same") or in group decisions (suppose the three group members have $a > b > c$, $b > c > a$, and $c > a > b$ and that they decide on each pair by majority vote).

Model 2: Suppose we observe a binary relation, $\succeq$, for any two elements of X. We say that $a \succeq b$ if a is at least as good as b. For consistency, we require:

- *Completeness*: At least one of $a \succeq b$ and $b \succeq a$ holds.
- *Transitivity*: If $a \succeq b$ and $b \succeq c$ then $a \succeq c$.

In this description of preferences, we say that $a \succ b$ if $a \succeq b$ and $b \not\succeq a$. We say that $a \sim b$ if $a \succeq b$ and $b \succeq a$.

Theorem 1.2 *The two models of preferences are isomorphic.*

Proof Given a consistent set of function responses, we map them to a consistent binary relation with the same meaning. To do this, we set $a \succeq a$ for all $a \in X$. If $a \neq b$, then we know that $f(a,b) = f(b,a)$ and it is enough to consider only one of them. We then map $(a > b)$ to $a \succeq b$ and $b \not\succeq a$, I to $a \succeq b$ and $b \succeq a$, and $b > a$ to $a \not\succeq b$ and $b \succeq a$. This mapping is:

- Well-defined, because $f(a,b) = f(b,a)$ and $f(a,b)$ is defined for all a, b.
- Complete: Note that $a \succeq a$ by definition, and at least one of $a \succeq b$ or $b \succeq a$ for any of the three possibilities for $f(a,b)$.
- Transitivity: Suppose that $X \succeq y$ and $y \succeq z$. Then, $f(x,y) \in \{x > y, I\}$. and $f(y,z) \in \{y > z, I\}$. By transitivity in f, $f(x,z) \in \{x > z, I\}$, so $x \succeq z$.
- One-to-one: Suppose $f_1(a,b) \neq f_2(a,b)$. Then at least one of the elements of the binary relations disagrees because the mappings will differ.
- Onto: We may invert this mapping, and the results will be a consistent set of preferences in function terms ($f(a,b) = f(b,a)$ since both depend on whether $a \succeq b$ and $b \succeq a$; transitivity in one maps to transitivity to the other). $\blacksquare$

To describe preferences, one must identify X. Preferences can be defined either as a finite list or as a function that takes the characteristics of two options and returns a decision based on them

Definition Suppose a *utility function*, $U : X \to R$, is given. We define a preference relation, $\succeq$, on X by $x \succeq y$ if and only if $U(x) \geq U(y)$.

Note that any preferences based on a utility function satisfy completeness and transitivity because the real numbers satisfy completeness and transitivity.

Proposition 1.3 *Suppose $U : X \to R$ is a utility function representing preferences, $\succeq$. Let $f : R \to R$ be any strictly increasing function. Then, $f \circ U$ represents the same preference relation.*

Proof

$$\begin{aligned} f(U(x)) \geq f(U(y)) &\Leftrightarrow U(x) \geq U(y) \\ &\Leftrightarrow x \succeq y \end{aligned}$$

$\blacksquare$

Lemma 1.4 *If A is a finite set with a preference relation, $\succeq$, then there exists $a^* \in A$ such that $a \succeq a^*$ for all $a \in A$. (We call a^* a* minimal element.*)*

Proof If A has one element, then that element is minimal. Suppose that any set of $n-1$ elements has a minimal element. Suppose A has n elements. Then, $A = \{a\} \cup A'$, where A' has $n-1$ elements. Let m be the minimal element of A'. If $a \succeq m$, then m is the minimal element of A. Otherwise, by completeness, $m \succeq a$. By transitivity, for any $b \in A$, $b \succeq m \succeq a$, and a is minimal. ∎

Proposition 1.5 *Suppose X is a finite set with preferences $\succeq$. Then, there is a utility function, U, that represents these preferences.*

Proof Let X_1 be the set of minimal elements (there may be more than one minimal element, because of indifference). In general, let X_k be the set of minimal elements in $X - (X_1 \cup ... \cup X_{k-1}$. This creates a finite partition of X. For each $x \in X_k$, define $U(x) = k$. This is a well-defined function because we have a partition. If $a \sim b$ then a and b are in the same partition (minimal at the same time) and $U(a) = U(b)$. If $a \succ b$, then a is removed before b, and $U(a) < U(b)$. ∎

Definition We call the equivalence classes of a preference relation on X *indifference sets.*

Proposition 1.6 *If X is countable, then any preference relation on X can be represented by a utility function on a bounded range.*

Proof Since X is countable, we may write $X = \{a_1, a_2, a_3, ...\}$. Define $U(a_1) = 0$. Suppose we have defined $U(a_1), ..., U(a_n)$ such that $a_i \succ a_j$ if and only if $U(a_i) \geq U(a_j)$. We define $U(a_{n+1})$. If $a_{n+1} \sim a_i$ for $i \leq n$, then define $U(a_{n+1}) = U(a_i)$. Otherwise, we may define a partition $\{a_1, ..., a_n\} = B \cup C$, where $B = \{a_i : x \succeq a_i\}$ and $C = \{a_i : a_i \succeq x\}$. Then, any element of $\{U(b_1), ..., U(b_k), -1\}$ is strictly greater than any element of $\{U(c_1), ..., U(c_{n-k}), 1\}$. Therefore, there is some number, r, that lies below all the numbers in the first set and above all the numbers in the second set. Define $U(a_{n+1}) = r$. This process yields a utility function with all values lying strictly between -1 and 1. ∎

Definition Suppose we have a vector of n different preference relations, $\succeq_1, ..., \succeq_n$, for an item, x. The *lexicographic preferences* are defined by $x \succeq_L y$ if there is some k such that $x \sim_j y$ for all $j < k$ and $x \succ_k y$ or $x \sim_k y$ for all $k = 1, ..., n$.

Proposition 1.7 *The lexicographic preferences on* $[0, 1] \times [0, 1]$ *based on the first coordinate and then the second coordinate cannot be represented by a utility function.*

Proof Suppose such a utility function, U, exists. Choose $a \in [0, 1]$. Then, $(a, 1) \succ_L (a, 0)$ and $U(a, 1) > U(a, 0)$. There must be some rational number,

$q_a \in Q$, in $[U(a,0), U(a,1)]$. We may find such a rational number fo every $a \in [0,1]$. Note that for any $a \neq b$, $U(a,0) < q_a < U(a,1) < U(b,0) < q_b < u(b,a)$, and this is a one-to-one function, $f : [0,1] \to Q$. This contradicts the countability of Q. ∎

1.1 Preferences in Euclidean Space

When not otherwise stated, assume that $X \subset R^n$.

Definition A preference relation, $\succeq$, is *continuous* if whenever $x_0 \succ y_0$ there are ϵ-balls around x and y such that for all $x \in B_\epsilon(x_0)$ and $y \in B_\epsilon(y_0)$ we have $x \succ y$.

Definition The *graph* of a binary relation, $\succeq$, is defined by $G(\succeq) = \{(x,y) : x \succeq y\} \subseteq X \times X$.

Definition A preference relation, $\succeq$, is *continuous* if $G(\succeq)$ is a closed set in $X \times X$. (That is, if $\{(x_n, y_n)\} \subset G(\succeq)$ is a sequence that converges in $X \times X$ to (x^*, y^*), then $(x^*, y^*) \in G(\succeq)$.)

Theorem 1.8 *The two definitions of continuity are equivalent.*

Proof ($\Rightarrow$) Suppose that whenever $x \succ y$, there are ϵ-balls around x and y with every element of $B_\epsilon(x)$ preferred to every element of $B_\epsilon(y)$. Let $\{(x_n, y_n)\} \subset G(\succeq)$ converge to (x^*, y^*). Suppose $x^* \prec y^*$. Take ϵ-balls around x^* and y^* such that any element of $B_\epsilon(x^*)$ is strictly not preferred to $B_\epsilon(y^*)$. But, by the definition of convergence, $x_n \in B_\epsilon(x^*)$ and $y_n \in B_\epsilon(y^*)$, so that $x_n \prec y_n$. This contradicts $(x_n, y_n) \in G(\succeq)$.

($\Leftarrow$) Suppose that $G(\succeq)$ is closed. Suppose there exists a, b such that $a \prec b$ but, for all $\epsilon > 0$ not all points of $B_\epsilon(b)$ are preferred to $B_\epsilon(a)$. For each n, choose (a_n, b_n) such that $a_n \in B_{1/n}(a)$, $b_n \in B_{1/n}(b)$, and $a_n \succeq b_n$. Then, (a_n, b_n) converges to (a, b) and is contained in $G(\succeq)$. Then, we must have $a \succeq b$. This is a contradiction. ∎

Proposition 1.9 *If a utility function, U, is continuous, then the preference relation it describes is also continuous.*

Proof Suppose $U(a) > U(b)$. Then, there are δ-balls around a and b such that $U(x) > U(a) - \frac{U(a)-U(b)}{2}$ for all $x \in B_\delta(a)$ and such that $U(y) < U(b) + \frac{U(a)-U(b)}{2}$ for all $y \in B_\delta(b)$. Thus, the preferences are continuous. ∎

Lemma 1.10 *Suppose that preferences are continuous, that X is convex (or, at least, connected), and that $x \succ y$. Then, there exists z on the line that connects x and y such that $x \succ z \succ y$.*

Proof Let $x_1 = x, y_1 = y$. Suppose there is no z such that $x \succ z \succ y$. Then, the point halfway between x_1 and y_1 satisfies either $z \succeq x_1$ or $z \preceq y_1$. In the former case, set $x_2 = z, y_2 = y_1$; in the latter case, set $x_2 = x_1, y_2 = z$. For the n^{th} elements of the sequence, choose z halfway between x_{n-1} and y_{n-1}. Set $x_n = z, y_n = y_{n-1}$ if $z \succeq x_{n-1}$ and $x_n = x_{n-1}, y_n = z$ if $z \preceq y_{n-1}$. Then, the sequence $\{(x_n, y_n)\}$ has $x_n \succeq x$ and $y_n \preceq y$ for all n and they converge to z^* between x and y. By continuity, $z^* \succeq x$ and $z^* \preceq y$. By transitivity, $y \succeq x$. This is a contradiction. ∎

Theorem 1.11 Debreu. *If preferences are continuous on $X \subset R^n$ then there is a continuous utility function that represents the preferences.*

Proof Assume $X \subset R^n$ is convex and that preferences, $\succeq$, are continuous. Assume that X contains a countable dense set, Y. Since Y is countable, the preference relation induces preferences on Y that can be represented by a utility function, $V : Y \to [-1, 1]$. Suppose $x \in X$. Define $U(x) = \sup\{V(z) : z \in Y, x \succ z\}$. Define $U(x) = -1$ if $\{z \in Y : x \succ z\} = \emptyset$ (this is the case in which x must be a minimal point, since all of Y is better than x and Y is dense). Note that $U(x) \in [-1, 1]$ for all x. We show that this is a utility function representing these preferences.

Case $x \sim y$: $U(x) = U(y)$ because $z \prec x$ if and only if $z \prec y$ and we must be taking the supremum over the same set.

Case $x \succ y$: If $x \succ y$ then there exists a such that $x \succ a \succ y$. Note that a does not need to be an element of Y. However, there is a ball, $B_\epsilon(a)$ such that every element in the ball is better than y and worse than x. Because Y is dense in X, we may choose $a' \in Y \cap B_\epsilon(a)$, and then $x \succ a' \succ y$. We may repeat this process to choose $b \in Y$ such that $x \succ a' \succ b \succ y$. Then, $U(x) \geq V(a') > V(b) \geq U(y)$, and we have strict inequality. ∎

Note that U in this construction need not be continuous (because there may be countably many jumps) or differentiable.

1.2 Consumer Preferences

In the case of the consumer, we take $X = R^K_+$, so that the consumer is choosing among bundles with different (non-negative) amounts of K commodities.

Definition A preference relation, $\succeq$, on X, satisfies *monotonicity* if, for all $x, y \in X$, (1) if $x_k \geq y_k$ for all k, then $x \succeq y$ and (2) if $x_k > y_k$ for all k, then $x \succ y$.

Definition A preference relation, $\succeq$, on X, satisfies *strong monotonicity* if for all $x, y \in X$, if $x_k \geq y_k$ for all k and $x_k \neq y_k$, then $x \succ y$.

Theorem 1.12 Debreu. *Any consumer preference relation satisfying monotonicity and continuity can be represented by a utility function.*

Proof Let x be any bundle. Then, by monotonicity, $(0, ..., 0) \prec x \prec (\max(x_k), ..., \max(x_k))$. By continuity, there must be a bundle on the line connecting $(0, ..., 0)$ and $(\max(x_k), ..., \max(x_k))$ which is indifferent to x. Since the bundle must be on the main diagonal, we have $x \sim (u(x), ..., u(x))$. Let $u(x)$ be the utility function. This must represent the preferences, since (by transitivity) $x \succeq y$ if any only if $(u(x), ..., u(x)) \succeq (u(y), ..., u(y))$, which, by monotonicity, happens if and only if $u(x) \geq u(y)$.

Definition A preference relation, $\succeq$, on X, satisfies *convexity* if for all $x \succeq y$ and $\alpha \in (0, 1)$, $\alpha x + (1 - \alpha)y \succeq y$.

Definition A set A is convex if for all $a, b \in A$ and $\lambda \in [0, 1]$, $\lambda a + (1 - \lambda)b \in A$.

Definition A preference relation, $\succeq$, on X, satisfies *convexity* if for all $y \in X$, the set $AsGood(y) = \{z \in X | z \geq y\}$ is convex.

Proposition 1.13 *The two definitions are equivalent.*

Proof ($\Rightarrow$). Suppose $\succeq$ satisfies the first definition. Let $a \succeq y$ and $b \succeq y$. Then, for any $\lambda \in [0, 1]$, $\lambda a + (1 - \lambda)b \succeq b \succeq y$, and $\lambda a + (1 - \lambda)b \in AsGood(y)$.

($\Leftarrow$). Suppose $\succeq$ satisfies the second definition. If $x \succeq y$ then $x, y \in AsGood(y)$. Thus, $\alpha x + (1 - \alpha)y \in AsGood(y)$, and $\alpha x + (1 - \alpha)y \succeq y$. ∎

Definition A preference relation, $\succeq$, on X, satisfies *strict convexity* if for every $a \succeq y$ and $b \succeq y$, $a \neq b$, and $\lambda \in (0, 1)$, $\lambda a + (1 - \lambda)b \succ y$.

Definition A function, u, is *quasi-concave* if for all y, the set $\{x | u(x) \geq u(y)\}$ is convex.

Proposition 1.14 *A preference relation is convex if and only if its corresponding utility function is quasi-concave. (Note that the utility function does not need to be concave, only quasi-concave.)*

Definition A preference, $\succeq$, is *homothetic* if $x \succeq y$ implies that $\alpha x \succeq \alpha y$ for all $\alpha \geq 0$.

Definition A function, u, is *homogenous of degree* λ if $u(\alpha x) = \alpha^\lambda u(x)$.

Proposition 1.15 *Any preference relation represented by a utility function that is homogenous of any degree λ is homothetic.*

The lexicographic preferences are also homothetic.

Theorem 1.16 *Any homothetic, continuous, and increasing preference relation on the commodity bundle space can be represented by a utility function that is homogenous of degree one.*

Proof Recall that for any $x \in X$, $x \sim (u(x), ..., u(x))$ for some $u(x)$, and that $u(x)$ represents $\succeq$. Since the preferences are homothetic, $\alpha x \sim (\alpha u(x), ..., \alpha u(x))$, so that $u(\alpha x) = \alpha u(x)$. ∎

Definition A preference is *quasi-linear* in commodity 1 (called the *numeraire*, and sometimes allowed to be negative) if $x \succeq y$ implies that $(x+\epsilon e_1) \succeq (y+\epsilon e_1)$, where $e_1 = (1, 0, ..., 0)$ and $\epsilon > 0$.

The indifference curves of quasi-linear preferences are parallel (relative to the axis of the quasi-linear commodity).

Theorem 1.17 *Any continuous preference relation satisfying strong monotonicity in commodity 1 and quasi-linearity in commodity 1 can be represented by a utility function of the form* $x_1 + v(x_2, ., , , .x_K)$.

Proof In this case, for every $(x_2, ..., x_K)$ there is some number $v(x_2, ..., x_K)$ such that $(v(x_2, ..., x_K), 0, ..., 0) \sim (0, x_2, ..., x_K)$ (get details of proof from solutions?). Then, by quasi-linearity, $(x_1 + v(x_2, ..., x_K), 0, ..., 0) \sim (x_1, x_2, ..., x_K)$. Because the preferences are strongly monotonic in the first commodity, $x_1 + v(x_2, ..., x_K)$ represents the preferences.

Definition Suppose $\succeq$ satisfy monotonicity and convexity. For $x \in X$, we say that $d \in R^K$ is an *improvement direction* if there is some $\epsilon > 0$ such that $x + \epsilon d \succ x$ (note that this must also hold for any $\delta < \epsilon$ by convexity). Let $D(x)$ be the set of improvement directions (which includes any positive direction, by monotonicity). We say that $\succeq$ are *differentiable at the bundle* x if there is a vector $v(x)$ such that $d'v(x) > 0$ if and only if $d \in D(x)$. $v(x)$ is the vector of *subjective values* of the commodities. The preferences $\succeq$ are *differentiable* if they are differentiable at every bundle, x.

Proposition 1.18 *Suppose u is a differentiable, quasi-concave utility function. If all the vectors* $(\frac{du}{dx_1}(x), ..., \frac{du}{dx_K}(x))$ *are nonzero, then the induced preference relation is differentiable with* $v_k(x) = \frac{du}{dx_k}(x)$.

2 Choice

Definition A *choice function* is a mapping from a subset, A, of a set X to an element in that set, $C(A) = a \in A$. That is, $c : D \to X$. The domain, D, of this function does not include the empty set, and need not include all subsets of X.

The choice function describes the behavior of an individuals when give a choice from the set. It need not reflect preferences directly. Under rational behavior, with preferences $\succeq$, we assume that $C_{\succeq}(A)$ is the maximal element in A. That is, $C_{\succeq}(A) \succeq a$ for all $a \in A$. With this choice function, we do not allow for indifference between two options.

Proposition 2.1 *Suppose we observe a choice function, C, on a domain, D, of X that contains all the subsets of size 2 and 3. Suppose that for all $A, B \in D$ with $A \subseteq B$ and $C(B) \in A$, $C(B) = C(A)$. Then, we may attach a set of preferences such that all choices maximize this set of preferences. (We call such a choice function* rationalizable.*)*

Proof For any $a, b \in X$, consider $C(\{a,b\})$. We define $a \succeq b$ if $C(\{a,b\}) = a$. This is complete, since one of the two options must be chosen. Furthermore, transitivity holds: If $C(\{a,b\}) = a$, $C(\{b,c\}) = b$, and $C(\{a,c\}) = c$, then we consider $C(\{a,b,c\})$. Since all three sets are contained in $\{a,b,c\}$, no element can be chosen and we reach a contraction of the assumption.

Suppose that this preference relation does not map to the choice function. Then, there exists x, X such that $x \in A$ but $C(A) \not\succeq x$. Then, $C(\{x, C(A)\}) \neq C(A)$, which contradicts the assumption, since $\{x, C(A)\} \subset A$. ∎

Definition Given a set X, a set of non-empty subsets of X, D, we define a more general choice function by $C(A) \subseteq A$, $C(A) \neq \emptyset$ for all $A \in D$. That is, this is the set of equally maximal elements in A.

Axiom 2.2 Weak Axiom. *Given $A, B \in D$ and $a, b \in A \cap B$, if $a \in C(A)$ and $b \in C(B)$ then $a \in C(B)$.*

Theorem 2.3 *C satisfies the weak axiom if and only if there exist preferences, $\succeq$, such that $C(A) = \{x | x \succeq a$ for all $a \in A\}$.*

Proof ($\Leftarrow$). Suppose $a \in C(A)$ and $b \in C(B)$. Then, $a \sim b$ and a must also be a maximizer in the set B.

($\Rightarrow$). Let a, b be given. Define $a \succeq b$ if $a \in C(\{a,b\})$. We show that this is a preference relation. First, it is complete, because $C(\{a,b\}) \neq \emptyset$, so $a \succeq b$, $b \succeq a$, or both. For transitivity, suppose $a \succeq b$, $b \succeq c$, but $a \not\succeq c$. Then, we must have $C(\{a,b,c\}) = \emptyset$, which is impossible.

Suppose this is not the correct preference relation. Then, $C(B) \neq C_{\succeq}(B)$. If there is some $x \in C(B)$ with $x \notin C_{\succeq}(B)$, then there exists $y \in C_{\succeq}(B)$ with $y \succeq x$. Then, $C(\{x,y\}) = \{y\}$, which contradicts the weak axiom. Suppose there is some $x \in C_{\succeq}(B)$ and $x \notin C(B)$. Choose any $y \in C(B)$. By the weak axiom, $x \notin C(\{x,y\})$, so $C(\{x,y\}) = \{y\}$, meaning that $y \succ x$. This contradicts $x \in C_{\succeq}(b)$. Thus, we must have $C(B) = C_{\succeq}(b)$. ∎

2.1 Consumer Choice

Definition A *budget set*, $B(p,w)$, for $p \in R_+^K, w \in R_+$ is defined as $B(p,w) = \{x \in R^k | px \leq w\}$. This is the set of all bundles that can be purchased if the price of good k is p_k with the wealth endowment, w.

The following are true about budget sets:

- $B(p,w)$ is closed because it is defined by $K+1$ weak inequalities, $x_j \geq 0$ for $j = 1, ..., K$ and $\sum_{j=1}^K p_j x_j \leq w$.
- For all p, w, $B(p,w)$ is compact, since it is closed and bounded.
- $B(p,w)$ is non-empty (it always contains zero).
- $B(p,w)$ is convex: If $x, y \in B(p,w)$, then $\lambda x + (1-\lambda) y \in B(p,w)$ since $\lambda px + (1-\lambda) py \leq w$.

Definition The *consumer's problem* is to find a bundle that maximizes their preferences over the budget set. That is, $P(p, w^*) = \max_x\{u(x)|px \leq w^*\}$.

Proposition 2.4 *If preferences, $\succeq$, are continuous, then there is at least one point that maximizes $\succeq$ over $B(p, w)$.*

Proof (Non-utility proof.) Suppose there is some $B(p, w)$ with no maximum for $\succeq$. Define $Inferior(z) = \{y|z \succ y\}$. No point in $B(p, w)$ is a maximum, so for all $x \in B(p, w)$, $x \in Inferior(z)$ for some z. Thus, $\{Inferior(z)|z \in B(p, w)\}$ is an infinite open cover for $B(p, w)$. Since $B(p, w)$ is compact, there is a finite subcover, $\{Inferior(z_1), ..., Inferior(z_L)\}$. Thus, every element of $B(p, w)$ is strictly inferior to at least one of $z_1, ..., z_L$. By transitivity, $z^* = \max(z_1, ..., z_L)$ cannot be inferior to any of the other elements. This is a contradiction, so there must be a maximum. $\blacksquare$

Proposition 2.5 *If $\succeq$ are continuous and strictly convex, then there is a unique maximum in the budget set, $B(p, w)$.*

Proof Suppose there are two maximizers, y, z. Then, $y \succeq z$, so $\lambda y+(1-\lambda)z \succ z$ for any $\lambda \in (0, 1)$ by strict convexity. Furthermore, $\lambda y + (1 - \lambda)z \in B(p, w)$, so this point is feasible and strictly preferable to the two maximizers. This is a contradiction. $\blacksquare$

Proposition 2.6 *If $\succeq$ are continuous, strictly convex, and monotonic, then the solution, x^*, satisfies $px^* = w$.*

Proof Suppose not. Then, $px^* < w$ and there exists $\epsilon > 0$ such that $p(x_1^* + \epsilon, ..., x_K^* + \epsilon) \leq w$. By monotonicity, this point is strictly preferable, which is a contradiction. $\blacksquare$

Proposition 2.7 *Suppose $\succeq$ are monotonic, convex, and differentiable. Suppose x^* solves the consumer's problem. Then, $\frac{v_k(x^*)}{p_k} \geq \frac{v_j(x^*)}{p_j}$ for all k such that $x_k^* > 0$ and for all j. (If $x_k^* > 0$ for all k, then all of the $\frac{v_k(x^*)}{p_k}$ must be equal.)*

Proof Suppose not. Then, there exists k, l such that $x_k^* > 0$ but $\frac{v_k(x^*)}{p_k} < \frac{v_l(x^*)}{p_l}$. We define a feasible direction of improvement by $d_k = -1$, $d_l = \frac{p_k}{p_l}$, $d_i = 0$ otherwise. This is feasible since $x_k^* > 0$ and $dp = 0$. Furthermore, $dv(x^*) = -1(v_k(x^*)) + \frac{p_k}{p_l}v_l(x^*) > 0$. Thus, we have a direction of improvement. $\blacksquare$

Proposition 2.8 *Suppose $\succeq$ are monotonic, convex, and differentiable. Sufficient conditions for x^* to be optimal are: (1) $px^* = w$ and (2) $\frac{v_k(x^*)}{p_k} \geq \frac{v_j(x^*)}{p_j}$ for all j, k where $x_k^* > 0$.*

Proof Suppose x^* is not a solution. Then, there is some z such that $pz \leq px^*$ and $z \succ x$. By continuity and monotonicity, there is some $y \neq z$ with $y_k \leq z_k$, $y \succ x$, and $py < pz \leq px^*$. By convexity, any small move in the direction $y - x^*$

is an improvement. By differentiability, $v(x^*)(y - x^*) > 0$. Let $\mu = \frac{v_k(x^*)}{p_k}$ for all k with $x_k^* > 0$. Then,

$$0 > p(y - x^*) = \sum p_k(y - x_k^*) \geq \sum v_k(x^*)(y_k - x_k^*)/\mu$$

(Either $p_k = \frac{v_k(x^*)}{\mu}$ or $p_k \geq \frac{v_k(x^*)}{\mu}$ and $x_k^* = 0$.) This implies that $0 \geq v(x^*)(y - x^*)$, which contradicts that y is an improvement direction. ∎

(In the case of two goods and an interior optimum, the slope of the indifference curve through the optimal point is always $-\frac{p_1}{p_2}$.)

3 Demand Functions

Definition The demand function, $x : R_+^K \times R_+ \to R_+^K$ is given by $x(p, w)$ equal to the choice of the consumer from $B(p, w)$.

Some properties of the demand function:

- Homogeneity of degree 0: $x(p, w) = x(\lambda p, \lambda w)$, since $B(p, w) = B(\lambda p, \lambda w)$.
- *Walras's Law*: If preferences are monotonic, then for all p, w, $px(p, w) = w$.

Proposition 3.1 *$x(p, w)$ is continuous in p if preferences are continuous and $x(p, w)$ is well-defined.*

Proof (Non-utility version.) Suppose not. Then, there is some sequence $\{p_n\}$ that converges to p^* such that $\{x(p_n, w)\}$ does not converge to $x(p^*, w)$. Note that each $x_n \in B(p_n, w)$. Since the p_n are bounded away from zero (because $p^* > 0$), each x_{nk} is bounded above, and all the $\{x_n\}$ lie in a bounded set. Thus, there must be a subsequence, $\{x_{n_j}\}$ that converges to some y^*. Since $p_n x_n \leq w$, $p^* y^* \leq w$. If $x^* \neq y^*$, we must have $y^* \prec x^*$ (because the maximizer is unique). By continuity, there must be a point z in the interior of the budget set with $y^* \prec z \prec x^*$. Thus, $z \in B(p_n, x)$ for large enough n but is preferable to y^* and is therefore preferable to $x(p^n, w)$. This is a contradiction. ∎

Since the domain of budget sets does not contain sets with exactly two or three elements, the previous version of rationalizability will not work.

Definition A demand function, $x(p, w)$ is *fully rationalizable* if there is some preference relation, $\succeq$, such that $x(p, w)$ is the unique maximizer of $\succeq$ over $B(p, w)$ for any p, w.

Definition A demand function, $x(p, w)$, is *rationalizable* if there is some preference relation, $\succeq$, that satisfies monotonicity, such that $x(p, w)$ is a maximizer of $\succeq$ over $B(p, w)$ for any p, w.

The fastest way to show that a demand function is rationalizable is usually to find a utility function that induces that demand function.

The weak axiom is necessary but not sufficient to show rationalizability (which makes it helpful for showing that some preferences are not rationalizable). It does not always help, because the union of two budget sets is not necessarily a budget set.

Axiom 3.2 Strong axiom of revealed preference. *Suppose we have* $(p_1, w_1), ..., (p_N, w_N)$ *with* $x(p_i, w_i) \neq x(p_{i+1}, w_{i+1})$ *and* $p_i x(p_{i+1}, w_{i+1}) \leq w_1$ *for* $i = 1, ..., N-1$*. Then,* $p_N x(p_1, w_1) > w_N$*. (That is, if one keeps choosing* $x(p_i, w_i)$ *over* $x(p_{i+1}, w_{i+1})$*, one cannot then choose* $x(p_1, w_1)$ *over* $x(p_N, w_N)$ *because it will not be feasible.)*

Proposition 3.3 *Suppose preferences satisfy Walras's Law and the Weak Axiom. Suppose* $w' = p'x(p, w)$*. Then, either* $x(p', w') = x(p, w)$ *or* $(p'-p)(x(p', w')-x(p, w)) < 0$*.*

Proof

$$\begin{aligned}(x(p,w) - x(p',w'))(p - p') &= px(p,w) - p(x'w') - p'x(p,w) + p'x(p',w') \\ &= w - px(p',w') - w' + w' \\ &= w - px(p',w')\end{aligned}$$

This is positive if $w \geq px(p', w')$; this cannot happen under the weak axiom. Thus, the first expression must be negative unless $x(p, w) = x(p', w')$. ∎

Corollary 3.4 *If* p' *increases the price of one good, then* $p - p' = (0, ..., 0, -\epsilon, 0, ..., 0)$*. In this case, we must have* $0 > -\epsilon(x_j(p, w) - x_j(p', w'))$*, and* $x_j(p, w) > x_j(p', w')$*. (This is the correct form of the Law of Demand; it is a law of compensated demand.)*

The traditional *Law of Demand* that states that increasing p_j and holding everything else fixed necessarily leads to a decrease in $x_j(p, w)$ does not necessarily hold. (For example, there may be Giffen goods.)

3.1 Indirect Preferences

Definition Suppose there is a preference relation, $\succeq$, on a set X. Let D be the domain of the choice function. We define the *indirect preferences*, $\succeq^*$ on D, by $A \succeq^* B$ if $C(A) \succeq C(B)$ for any $A, B \in D$.

Definition If u represents the preferences, $\succeq$, and the choice function is well-defined, then we define the *indirect utility function* by $v(A) = u(C(A))$. The indirect utility function represents $\succeq^*$.

Consider the indirect preferences, $\succeq^*$ on budget sets, defined by $(p, w) \succeq^* (p', w')$ when $x(p, w) \succeq x(p', w')$. The following properties hold:

- $(\lambda p, \lambda w) \sim^* (p, w)$.
- $\succeq^*$ is non-increasing in p_k and increasing in w.
- If the preference relation $\succeq$ is continuous, then $\succeq^*$ is continuous as well (and so is its indirect utility function).
- If $(p, w) \succeq^* (p', w')$ then $(p, w) \succeq^* (\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w')$ for all $\lambda \in (0, 1)$. Equivalently, the indirect utility function is quasi-convex. (PROOF?)

Theorem 3.5 Roy's Inequality. *Assume that the demand function satisfies Walras's Law. Let $H = \{(p, w)|px(p^*, w^*) - w = 0\}$ for some (p^*, w^*). The hyperplane, H, is tangent to the $\succeq^*$ indifference curve through (p^*, w^*). Equivalently, if v is the indirect utility function, then $-(\frac{\partial v}{\partial p_k}(p^*, w^*))/(\frac{\partial v}{\partial w}(p^*, w^*)) = x_k(p^*, w^*)$.*

Proof $(p^*, w^*) \in H$. For any $(p, w) \in H$, $x(p^*, w^*) \in B(p, w)$, so $x(p, w) \succeq x^*(p, w)$, so that $(p, w) \succeq^* (p^*, w^*)$, and the hyperplane must always lie above the indifference curve.

Suppose $\succeq^*$ is represented by a differentiable v. Then,

$$H = \{(p, w)|(\frac{\partial v}{\partial p_1}(p^*, w^*), ..., \frac{\partial v}{\partial p_K}(p^*, w^*), \frac{\partial v}{\partial w}(p^*, w^*))(p - p^*, w - w^*) = 0\}$$

Since $w^* = p^* x(p^*, w^*)$,

$$H = \{(p, w)|(x(p^*, w^*), -1)(p - p^*, w - w^*) = 0\}$$

Thus, $(\frac{\partial v}{\partial p_1}(p^*, w^*), ..., \frac{\partial v}{\partial p_K}(p^*, w^*), \frac{\partial v}{\partial w}(p^*, w^*))$ is proportional to $(x_1(p^*, w^*), ..., x_K(p^*, w^*), -1)$, and $-(\frac{\partial v}{\partial p_k}(p^*, w^*))/(\frac{\partial v}{\partial w}(p^*, w^*)) = x_k(p^*, w^*)$. ∎

3.2 The Dual Consumer Problem and Hicksian Demand

Definition The *dual consumer problem* is the problem of achieving a certain level of utility while minimizing the expenditure. That is, $D(p, u^*) = \min_x\{px|u(x) \geq u^*\}$, for a given u^*.

Proposition 3.6 *If x^* is the solution to $P(p, w^*)$, then it is also the solution to the dual problem, $D(p, u(x^*))$. If x^* is the solution to the dual problem $D(p, u^*)$ then it is also the solution to the problem $P(p, px^*)$.*

Proof Suppose x^* is not a solution to the dual problem $D(p, u(x^*))$ then there exists a strictly cheaper bundle, x, for which $u(x) \geq u(x^*)$. Then, there is some strictly positive vector, ϵ such that $p(x + \epsilon) < px^* \leq w$. By monotonicity, $u(x + \epsilon) > u(x) \geq u(x^*)$. This contradicts the assumption that x^* is a solution to $P(p, w^*)$.

Suppose x^* is not a solution to $P(p, w^*)$. Then there exists x such that $px \leq px^*$ and $u(x) > u(x^*) \geq u^*$. By continuity, there is a non-negative vector, ϵ with $u(x - \epsilon) > u^*$ and $p(x - \epsilon) < px^*$, which contradicts the assumption that x^* solves $D(p, u^*)$. ∎

Definition If $D(p, u)$ has a unique solution, we define $h(p, u)$ as the solution. This is called the *Hicksian demand function*.

Some properties of the Hicksian demand function:

- $h(\lambda p, u) = h(p, u)$.
- $h_k(p, u)$ is non-increasing in p_k. (Because both are optimal, $ph(p, u) \leq ph(p', u)$ and $p'h(p', u) \leq p'h(p, u)$, so that $(p - p')(h(p, u) - h(p', u)) \leq 0$; we then set $p - p' = (0, ..., \epsilon, ..., 0)$ and deduce that $h_k(p, u) - h_k(p', u) \leq 0$.)
- $h(p, u)$ is continuous is p.

Definition We define the *expenditure function* by $e(p, u) = ph(p, u)$. This is the amount of money needed to reach a certain level of utility.

Properties of the expenditure function:

- $e(\lambda p, u) = \lambda e(p, u)$.
- $e(p, u)$ is non-decreasing in p_k and strictly increasing in u.
- $e(p, u)$ is continuous in p (since $h(p, u)$ is continuous in p).
- $e(p, u)$ is concave in the vector p. (Let $x = h(\lambda p_1 + (1 - \lambda)p_2, u^*)$. Then, $e(p_1, u^*) \leq p_1 x, e(p_2, u^*) \leq p_2 x$. So, $e(\lambda p_1 + (1 - \lambda)p_2, u^*) = (\lambda p_1 + (1 - -\lambda)p_2)x \geq \lambda e(p_1, u^*) + (1 - \lambda)e(p_2, u^*)$.)

Proposition 3.7 Dual Roy's Equality. *The hyperplane,* $H = \{(p, e) | e = ph(p^*, u^*)\}$ *is tangent to the graph of the function* $e(p, u^*)$ *at the point* p^*. *Equivalently,* $\frac{\partial e}{\partial p_i}(p, u) = h_i(p, u)$. *(CHECK.)*

Proof Since $ph(p^*, u^*) \geq ph(p, u^*)$ for all price vectors, p, the hyperplane must lie on one side of the graph of $e = ph(p, u^*)$ and it must intersect the graph at the point, $(p^*, e(p^*, u^*))$. ∎

4 Producers

Definition The *technology* of a producer is a $Z \subset R^k$ such that

- $0 \in Z$: The producer can do nothing.
- $Z \cap R^K_+ = \{0\}$: In order to produce output, some inputs must be used.
- Free disposal: If $x \in Z$ and the point y has more input and less output (down and to the left), then $y \in Z$.
- Z is closed: There are no discontinuities in production.
- Z is convex: This is equivalent to decreasing returns to scale.

For any point $z \in Z$, the input commodities are those with $z_k < 0$ and the output commodities are those with $z_k > 0$. (We sometimes assume that output is bounded as well.)

Definition A simpler version of the technology assumes that each good can be used only as an input or an output. Then, Z is confined to certain quadrants, where each good takes on the correct sign. Then, the boundary of the set is called the *production function*.

Definition The *producer's problem* is to maximize profits, px, subject to the constraint that $x \in Z$.

Note that the producer's problem is equivalent to maximizing the utility function, $u(x) = px$, over the choice set, Z.

Theorem 4.1 *If Z is bounded above and strictly convex, there is a unique solution, $z(p)$, to the producer's problem. We call $z(p)$ the* supply function. *(Note that, since $z(p)$ includes the inputs, it is really the producer's demand function as well.)*

Definition The *profit function* is given by $\pi(p) = pz(p)$. This is equivalent to finding the highest hyperplane that is tangent to Z.

The solution to the producer's problem is like the solution to the dual consumer problem, so the same mathematical properties hold. (FILL THEM IN?)
Some facts about production:

- $z(\lambda p) = z(p)$.
- $z(p)$ is continuous.
- *The Law of Supply*: $(p - p')(z(p) - z(p')) \geq 0$. Specifically, if the price of an output goes up, then production of that output will increase. If the price of an input goes up, then less of that input will be used (it will be less negative).

Note that the producer's preferences are very restricted; we do not model any non-profit motives.

Definition Suppose a producer uses inputs $1, ..., L$ to produce outputs $L + 1, ..., K$. We define the *cost function*, $c(p, y)$, to be the minimal cost associated with the production of the output $y \in R_+^{K-L}$ given the input prices, $p \in R_+^L$. That is, $c(p, y) = \min_a \{pa | (-a, y) \in Z\}$.

5 Decision-Making Under Uncertainty

Definition Let a finite set of prizes (or consequences), Z, be given. A *lottery* is a probability measure, p, on Z, so that $0 \leq p(z)$ and $\sum_{z \in Z} p(z) = 1$. We define the set of all lotteries on Z as $L(Z)$.

If $|Z| = K$, the set of lotteries is a simplex in R^K. We define preferences, $\succeq$, on $L(Z)$.

Definition The *support* of a lottery, p, is the set of feasible prizes, $\{z \in Z | p(z) > 0\}$.

Definition For any $z \in Z$, $[z]$ is the lottery that assigns probability 1 to z.

Definition A *compound lottery* is given by

$$\bigoplus_{k=1}^{K} \alpha_k p_k(z) = \sum_{k=1}^{K} \alpha_k p_k(z)$$

This is a two-stage lottery, where the first lottery determines which of the lotteries, $p_1, ..., p_K$, will happen at the second stage.

Axiom 5.1 Independence Axiom. *$p \succeq q$ if and only if, for any lottery, r, and $\alpha \in (0,1)$, $\alpha p \oplus (1-\alpha) r \succeq \alpha q \oplus (1-\alpha) r$.*

Proposition 5.2 *Suppose an agent has $\bigoplus_{k=1}^{K} \alpha_k p_k \succeq \bigoplus_{k=1}^{K} \alpha_k q_k$ if and only if $p_1 \succeq q_1$ and $p_k = q_k$ for $k > 1$. This property is equivalent to the independence axiom.*

Proof Consider the three-stage lotteries $\alpha_1 p_1 \oplus (1-\alpha_1)(\bigoplus_{k=2}^{K} \frac{\alpha_k}{1-\alpha_1} p_k)$ and $\alpha_1 q_1 \oplus (1-\alpha_1)(\bigoplus_{k=2}^{K} \frac{\alpha_k}{1-\alpha_1} q_k)$. These lotteries are equivalent to $\bigoplus_{k=1}^{K} \alpha_k p_k$ and $\bigoplus_{k=1}^{K} \alpha_k q_k$. By the independence axiom, it is sufficient for $p_1 \succeq q_1$.

Axiom 5.3 Continuity Axiom. *If $p \succ q$ then there are open balls about p, q such that every element of $B_\delta(p)$ is preferable to $B_\delta(q)$.*

Proposition 5.4 *The continuity axiom holds if and only if for any $p \succ q \succ r$ there exists $\alpha \in (0,1)$ such that $q \sim \alpha p \oplus (1-\alpha) r$.*

Proof *(Sketch.)* Notice that $1p \oplus 0r \succ q \succ 0p \oplus 1r$. Consider $\frac{1}{2}p \oplus \frac{1}{2}r$ and we may create a sequence of points that are strictly preferred and that are strictly not preferred to q. These sequences will converge to the same point. ∎

Proposition 5.5 *The two axioms do not imply each other.*

Proof Lexicographic preferences, in which $p(z_1)$ is used for the first comparison, then $p(z_2)$ is used for the second comparison (and so on) satisfy independence but not continuity. The preferences represented by $u(p) = \max(p_j)$ are continuous but do not satisfy independence. ∎

Definition Suppose we have a mapping of values, $v : Z \to R$ (these are called the *Von Neumann-Morgenstern utilities* or the *Bernoulli numbers*). Then, the *expected utility* of a lottery is given by $u(L) = E(v(z)) = \sum_{z \in Z} p(z) v(z)$. This is the *expected utility representation* of preferences over lotteries.

Proposition 5.6 *The expected utility representation satisfies the independence and continuity axioms.*

Proof Since $v(z)$ is fixed, $u(p)$ is linear in p, so the utility function is continuous.

Notice that:

$$\begin{aligned} u(\alpha p \oplus (1-\alpha)r) &= \sum_{z\in Z} v(z)(\alpha p(z) + (1-\alpha)r(z)) \\ &= \alpha \sum_{z\in Z} v(z)p(z) + (1-\alpha)\sum_{z\in Z} v(z)r(z) \\ &= \alpha u(p) + (1-\alpha)u(r) \\ u(\alpha q \oplus (1-\alpha)r) &= \alpha u(q) + (1-\alpha)u(r) \end{aligned}$$

Then, $u(\alpha p \oplus (1-\alpha)r) \geq u(\alpha q \oplus (1-\alpha)r)$ if and only if $u(p) \geq u(q)$, and the independence axiom holds. ∎

Lemma 5.7 *Suppose $x, y \in Z$ and $[x] \succ [y]$. Let $1 \geq \alpha > \beta \geq 0$. Then, if the preferences satisfy independence, $\alpha[x] \oplus (1-\alpha)[y] \succ \beta[x] \oplus (1-\beta)[y]$.*

Proof By independence, $[x] \succ [y]$ implies that $\alpha[x] \oplus (1-\alpha)[y] \succ \alpha[y] \oplus (1-\alpha)[y] = [y]$. Furthermore,

$$\begin{aligned} \alpha[x] \oplus (1-\alpha)[y] &= (1-\frac{\beta}{\alpha})(\alpha[x] \oplus (1-\alpha)[y]) \oplus \frac{\beta}{\alpha}(\alpha[x] \oplus (1-\alpha)[y]) \\ \beta[x] \oplus (1-\beta)[y] &= (1-\frac{\beta}{\alpha})[y] \oplus \frac{\beta}{\alpha}(\alpha[x] \oplus (1-\alpha)[y]) \end{aligned}$$

Thus, $\alpha[x] \oplus (1-\alpha)[y] \succ \beta[x] \oplus (1-\beta)[y]$. ∎

Theorem 5.8 *Any preference relation that satisfies independence and continuity has an expected utility representation.*

Proof Given Z, we may rank $[z_1], ..., [z_K]$. Choose $M, m \in Z$ such that $[M] \succeq [z] \succeq [m]$.

Case $[m] \sim [M]$: Let $p \in L(z)$. We may write $p = p(z_1)[z_1] \oplus ... \oplus p(z_K)[z_K]$. By independence, since $[m] \sim [z_i]$, $p \sim p(z_1)[m] \oplus ... \oplus p(z_k)[m] = [m]$. Thus, we are indifferent among all the lotteries. Let $v(z)$ be a constant. Then, $u(p)$ is constant.

Case $[M] \succ [m]$: Assign $v(M) = 1$ and $v(m) = 0$. For any $[M] \succ [z] \succ [m]$, by continuity, there is some α such that $[z]\ \alpha[M] \oplus (1-\alpha)[m]$. Define $v(z) = \alpha$. Let p be any lottery. Then, by independence,

$$\begin{aligned} p &\sim \bigoplus_{k=1}^{K} p(z_k)(v(z_k)[M] \oplus (1-v(z_k))[m]) \\ &= (\sum_{k=1}^{K} p(z_k)v(z_k))[M] \oplus (\sum_{k=1}^{K} p(z_k)(1-v(z_k)))[m] \end{aligned}$$

Then, $p \succeq q$ if and only if $\sum_{k=1}^{K} p(z_k)v(z_k) \geq \sum_{k=1}^{K} q(z_k)v(z_k)$. This is the expected utility function with values $v(z_k)$. ∎

Proposition 5.9 *Two value functions, v and w, represent the same preferences if and only if one is a positive affine transformation of the other.*

Proof Assuming that the preferences are not constant across all lotteries (in which case both value functions are just constants), we must have $v(M) > v(m)$ and $w(M) > w(m)$. There is a unique affine transformation such that $av(M) + b = w(M)$ and $av(m) + b = w(m)$. Then, for any $z \in Z$:

$$\begin{aligned} w(z) &= \alpha w(M) + (1 - \alpha) w(m) \\ v(z) &= \alpha v(M) + (1 - \alpha) v(m) \end{aligned}$$

This must satisfy $w(z) = av(z) + b$. ∎

For now, we set $Z = R$. We set $L(Z)$ be the set of lotteries with finite support. We also consider only value functions that are continuous and monotonic. That is if $t > s$, then $[t] \succ [s]$.

Definition Let p, q be lotteries. We say that p *stochastically dominates* q *of the first order*, or pD_1q, if for all expected utility preference relations satisfying monotonicity, $p \succeq q$.

This is a partial ordering on $L(Z)$; it is not complete, but it is transitive.

Definition For a lottery, p, and $x \in R$, we define $G(p, x) = P(\text{win } x \text{ or more}) = 1 - CDF(x)$.

Proposition 5.10 *pD_1q if and only if $G(p, x) \geq G(q, x)$ for all x.*

Proof Let a value function, v, and two lotteries, p, q, be given. Since both lotteries have finite support, the union of the supports is also finite. Let $x_1, ..., x_K$ be the union of the supports, and set $p(x_i) = 0$ if x_i was not in the support of p.

We may rewrite the expected utility as:

$$\begin{aligned} u(p) &= \sum_{k=1}^{K} p(x_k) v(x_k) \\ &= v(x_0) + (v(x_1) - v(x_0)) G(p, x_1) + ... + (v(x_K) - v(x_{K-1})) G(p, x_K) \\ &= \sum_{k=1}^{K} G(p, x_k)(v(x_k) - v(x_{k-1})) \end{aligned}$$

Then, $p \succeq q$ if and only if $\sum_{k=1}^{K} G(p, x_k)(v(x_k) - v(x_{k-1})) \geq \sum_{k=1}^{K} G(q, x_k)(v(x_k) - v(x_{k-1}))$.

For this to hold for an arbitrary value function, we must have $G(p, x_k) \geq G(q, x_k)$ for all x_k. ∎

5.1 Risk Aversion

Definition A preference relation, $\succeq$, if *risk-averse* if for all p, $[E(p)] \succeq [p]$.

Proposition 5.11 *A preference relation is risk-averse and satisfies expected utility maximization if and only if the value function, v, is concave.*

Proof Suppose not. Then, there exists a, b, λ such that $v(\lambda a + (1-\lambda)b) < \lambda v(A) + (1-\lambda)v(b)$. Then, $\lambda[a] \oplus (1-\lambda)[b] \succ [\lambda a + (1-\lambda)b]$, which violates risk aversion.

Suppose v is concave. Then, $v(\sum p(z)z) \geq \sum v(z)p(z)$ by Jensen's inequality. Thus, $[E(p)] \succ p$. ∎

Definition Let p be any lottery. Then, there exists some number, $CE(p)$ such that $p \sim [CE(p)]$. We call this the *certainty equivalent* of p.

Proposition 5.12 *A preference relation is risk averse if and only if $CE(p) \leq E(p)$ for all $p \in L(Z)$.*

Definition Preference relation, $\succeq_1$, is *more risk averse* than preference relation $\succeq_2$ if

- $p \succeq_1 [c]$ implies that $p \succeq_2 [c]$ for all $p \in L(Z), c \in Z$,
- for any $p \in L(Z)$, $CE_1(p) \leq CE_2(p)$, or
- we have $v_1 = \phi(v_2)$ with ϕ concave.

Proposition 5.13 *The three definitions of "more risk averse" are equivalent.*

Proof ($2 \Rightarrow 1$). Suppose $p \succeq_1 [c]$. Since $[p] \sim_1 [CE_1(p)]$, $CE_1(p) \geq c$. Since $CE_2(p) \geq CE_1(p)$ by assumption, $CE_2(p) \geq c$, and $[p] \succeq_2 [c]$.

($3 \Rightarrow 2$). Notice that $E(u_i(p)) = u_1(CE_i(p))$ for all p. Since u_i is strictly increasing, it is invertible, and $CE_i(p) = u_i^{-1}(E(u_i(p)))$. By assumption, $\phi = u_1(u_2^{-1})$ is concave, and we have:

$$\begin{aligned}
u_1(CE_2(p)) &= u_1(u_2^{-1}(E(u_2(p)))) \\
&= \phi(E(u_2(p))) \\
&\geq E(\phi(u_2(p))) \\
&= E(u_1(p)) \\
&= u_1(CE_1(p))
\end{aligned}$$

Taking u_1^{-1} of both sides shows that $CE_2(p) \geq CE_1(p)$.

($1 \Rightarrow 3$). Suppose $u_2(x) < u_2(y) < u_2(z)$. Then, we may choose $\lambda \in (0,1)$ such that $u_2(y) = \lambda u_2(x) + (1-\lambda)u_2(z)$. Taking ϕ of both sides of this equation yields $u_1(x) = \phi(\lambda u_2(x) + (1-\lambda)u_2(z))$. This equation is equivalent to $[y] \sim_2 \lambda[x] \oplus (1-\lambda)[z]$. Then, we must have $[y] \succeq_1 \lambda[x] \oplus (1-\lambda)[z]$, so that $u_1(y) \geq \lambda u_1(x) + (1-\lambda)u_1(z)$. Thus, $\phi(\lambda u_2(x) + (1-\lambda)u_2(z)) \geq \lambda u_1(x) + (1-\lambda)u_1(z)$, and ϕ is concave. ∎

Definition The *coefficient of absolute risk aversion* for a twice-differentiable value function is given by:

$$r(x) = -\frac{u''(x)}{u'(x)}$$

Note that this value is always positive (for a concave u) and may vary with x.

Proposition 5.14 *Preferences $\succeq_1$ are more risk averse than $\succeq_2$ if and only if $r_1(x) \geq r_2(x)$ for all x.*

Proof $\phi = u_1(u_2^{-1})$ is concave if and only if $\frac{d}{dx}u_1(u_2^{-1}(x)) = \frac{u1'(u_2^{-1}(x))}{u_2'(u_2^{-1}(x))}$ is non-increasing. Since u_2^{-1} is increasing, this can occur if and only if $\frac{u_1'(x)}{u_2'(x)}$ is non-increasing. Taking the logs and then the derivative, we find that this is equivalent to:

$$\begin{aligned} 0 &> \frac{d}{dx}(\log(u_1'(x)) - \log(u_2'(x))) \\ &= \frac{u_1'(x)}{u_1''(x)} - \frac{u_1'(x)}{u_1''(x)} \\ &= -\frac{1}{r_1(x)} + \frac{1}{r_2(x)} \end{aligned}$$

Thus, we must have $r_1(x) \geq r_2(x)$. ∎

In the case of lotteries of the form $p[x_1] \oplus (1-p)[x_2]$, for fixed p, with risk-averse preferences, we must have $[c] \succeq p[x_1] \oplus (1-p)[\frac{1}{1-p}(c - px_1)]$. If $\succeq$ is differentiable, then it must have slope $-\frac{p}{1-p}$ through any point, (c,c) on the main diagonal. If $\succeq_1$ is more risk averse than $\succeq_2$, then the indifference curve for $\succeq_1$ must lie inside (up and to the right) of the indifference curve for $\succeq_2$. In this case, note that $\phi''(x) = r(x)\frac{p}{(1-p)^2}$, and this yields another measure of risk aversion.