<u>Analysis</u>

Finite and Infinite Sets

Definition. An <u>initial segment</u> is $\{n \in \mathbf{N} \mid n \leq n_0\}$.

Definition. A <u>finite set</u> can be put into one-to-one correspondence with an initial segment. The empty set is also considered finite.

Definition. An <u>infinite set</u> is a set with no such bijection.

Definition. An infinite set is <u>countable</u> if it can be put into one-to-one correspondence with the natural numbers.

Definition. An infinite set is <u>uncountable</u> if no such bijections exists.

Technique. Show countability with a bijection.

Show uncountability by:

Showing a bijection onto an uncountable set.

Use contradiction.

Examples: the even natural numbers, the integers, and the rationals are all countable.

The real numbers and the irrationals are uncountable.

Theorem. A countable union of countable sets is countable.

The Real Numbers

1. **R** is a field.
   - Operations: +, *, their inverses
   - Abelian group under addition (closure, associativity, commutivity, zero, inverses)
   - Abelian group under multiplication (closure, associativity, commutivity, one, inverses)
   - Distibutivity
2. **R** is ordered.
   - Given $a, b \in \mathbf{R}$, $a \geq b$ or $a \leq b$.
   - **R** is a metric space.
3. **R** is complete.

   Definition. Let $S \subset \mathbf{R}$. $u \in \mathbf{R}$ is an <u>upper bound</u> for S if, for all $s \in S$, $s \leq u$. $u \in \mathbf{R}$ is a <u>lower bound</u> of S if, for all $s \in S$, $s \geq u$.

   Definition. An upper bound of S is the <u>least upper bound</u> (sup S or l.u.b. S) if it is less than or equal to any other upper bound of S. A lower bound of S is a <u>greatest lower bound</u> (inf S or g.l.b. S) if it is greater than or equal to any other lower bound of S.

   Remark. sup S and inf S might not be in S.

   Completeness Axiom. Every non-empty subset S of **R** that it bounded above has a least upper bound. Every non-empty set that is bounded below has a greatest lower bound.

Proposition. Real Numbers may be approximated by rationals.

Lemma. For any $x \in \mathbf{R}$ and positive integer N, there exists an integer n such that $n/N \leq x \leq (n+1)/N$.

Lemma. If $x \in \mathbf{R}$ and $\varepsilon > 0$, there exists a rational number r such that $|x - r| < \varepsilon$.

The Cantor Set

Definition. A sequence of intervals, I_n, is <u>nested</u> when the chain of inclusions $I_1 \supset I_2 \supset \ldots \supset I_n$ holds.

Theorem. For all $n \in \mathbf{N}$, let I_n be a non-empty, closed interval in **R**, $I_n = [a_n, b_n]$. Let $\{I_n\}$ be nested. Then there exists an element common to all the intervals.

Remark. If $\zeta = \sup \{a_n\}$ and $\eta = \inf \{b_n\}$ then $[\zeta, \eta]$ is the intersection of these intervals.

Definition. A <u>closed cell</u> in $\mathbf{R}^n$ is $\{x = (x_1, \ldots, x_n) \mid a_i \leq x_i \leq b_i \text{ for a fixed } a_i, b_i \in \mathbf{R}\}$.

Theorem. Any set of nested closed cells contains at least one point in its intersection.

Definition. The <u>Cantor Set</u> (*F*) may be constructed in the following way:

1. Begin with the unit interval. This is F_0.
2. For every interval in F_{n-1}, remove the open interval in the middle third. This is F_n.
3. F_n is the union of 2^n intervals of the form $[k/3^n, (k+1)/3^n]$, for certain k

Properties.

- The Cantor Set is non-empty.

- Some points are clearly the endpoints of intervals.
- Others are there, too (like ¼). These are the endpoints of intervals only in the limit.
- Can be put into 1-1 correspondence with **R** – uncountable.
- Equivalent to the set of all numbers between 0 and 1 that can be expressed with only 0's and 2's in ternary.
- Sparse. Every point is a boundary point.

Metric Spaces

Definition. A metric space is a set E together with a mapping d: E × E → **R**, which satisfies:
(1) d(p, q) ≥ 0 for all p, q (positive)
(2) d(p, q) = 0 if and only if p = q. (positive definite)
(3) d(p, q) = d(q, p) for all p, q (symmetry)
(4) d(p, r) ≤ d(p, q) + d(q, r) for all p, q, r ∈ E (triangle inequality)

Examples.
(1) **R**
(2) $\mathbf{R}^n$
(3) Any real vector space with an inner product.
(4) The "discrete metric" where d(p, q) = 0 if p = q, 1 otherwise.
(5) L_p, L_∞

Definition. L_∞ is the space of continuous functions on some set (ie. [0, 1]) with the metric d(f, g) = sup{|f(x)-g(x)| : x ∈ [0, 1]}.

Definition. L_p is the space of continuous functions on some set (usually [0, 1]) with the metric d(f, g) = $(\int|f(x)-g(x)|^p\, dx)^{1/p}$.

Definition. Let (E; d) be a metric space. Let p ∈ E and r ∈ **R**. Then, the open ball centered at p with radius r, denoted B(p; r), is defined by:

B(p, r) = {x ∈ E | d(p, x) < r}.

Definition. The closed ball centered at p with radius r is

B-bar(p; r) = {x ∈ E | d(p, x) ≤ r}.

Definition. A subset S of a metric space (E; d) is open if, for all p ∈ S, there is some open ball centered at p, contained entirely is S.

Definition. A subset S ⊂ (E; d) is closed if its complement is open.

Proposition. For any (E; d),
(1) ∅ is open.
(2) E is open.
(3) An arbitrary (possibly infinite) union of open sets is open.
(4) A finite intersection of open sets is open.

Corollary.
(1) ∅ is closed.
(2) E is closed.
(3) An arbitrary (possibly infinite) intersection of closed sets is closed.
(4) A finite union of closed sets is closed.

Definition. A subset S ⊂ E is bounded if it is contained in some ball in E.

Definition. If x ∈ E, a neighborhood of x is any set containing an open set that contains x.

Definition. An interior point of S ⊂ E is a point x ∈ S such that x has a neighborhood lying entirely in S. (Alternately, x lies in an open ball entirely contained in S.)

Definition. An exterior point of S is a point that has a neighborhood lying entirely outside S.

Definition. x is a boundary point of S if *every* neighborhood of x intersects S and S^C.

Proposition. A non-empty closed subset of **R** that is bounded above has a greatest element (in the set).

Proposition. Let S ⊂E. O is open in S if and only if O = O' ∩ S where O' is open in E.

Sequences

Definition. Let $\{p_n\}$ be a sequence of points in a metric space (E; d). Then p ∈ E is called the limit of the sequence if for all ε > 0 there exists N such that whenever n > N, $d(p_n, p) < \varepsilon$.

Remark. Some ways convergence may fail:

- Oscillation (never staying within an ε-ball).
- Heading toward "infinity."
- The limit is not in the space.

Proposition. A sequence of points $\{p_n\}$ in a metric space may have at most one limit.

Definition. Let $\{p_n\}$ be a sequence in (E; d). Let $n_1 < n_2 < \ldots$ be a strictly increasing sequence of positive integers. Then $\{p_{ni}\}$ is a <u>subsequence</u> of $\{p_n\}$.

Proposition. Let $\{p_n\}$ be a convergent sequence in (E; d). Then any subsequence of $\{p_n\}$ converges to the same limit.

Theorem. Let $S \subset$ (E; d). S is closed if and only if whenever $\{p_n\} \subset S$ converges in E, the limit lies in S.

Definition. A sequence of points $\{p_n\}$ in (E; d) is <u>Cauchy</u> if for any $\varepsilon > 0$, there exists N such that for all $n, m > N$, $d(p_n, p_m) < \varepsilon$.

Proposition. Any convergent sequence is Cauchy.

Completeness

Definition. A metric space (E; d) is <u>complete</u> if every Cauchy sequence in E converges to a point in E.

Proposition. A Cauchy sequence that has a convergent subsequence is itself convergent.

Remark. A convergent sequence is bounded.

Theorem. **R** with the standard metric is complete.

Definition. A sequence of numbers is <u>monotone</u> if it is either increasing or decreasing.

Proposition. A bounded monotone sequence (in **R**) converges. The limit is either the greatest lower bounded or the least upper bound.

Compactness

Definition. A collection, $G = \{G_\alpha\}$, of open sets is said to <u>cover</u> a set K if $K \subset \cup G_\alpha$. We call G an <u>open cover</u> of K.

Definition. $K \subset$ (E; d) is <u>compact</u> if every open cover of K has a finite subcover.

Definition. Let $S \subset$ (E; d). $p \in E$ is a <u>cluster point</u> of S if any open ball centered at p contains an infinite number of points of S.

Theorem (Bolzano-Weierstrass). An infinite subset S of a compact metric space (E; d) has at least one cluster point in E.

Corollary. Any sequence of points in a compact metric space has a convergent subsequence.

Corollary. A compact metric space is complete.

Theorem. Any compact subspace of a metric space is both closed and bounded.

Remark. Compact implies complete, closed and bounded. Complete implies closed. No other implications, though.

Theorem. If (E; d) is compact then a closed and bounded subset of E is compact.

Theorem (Heine-Borel). A subset of $\mathbf{R}^n$ with the standard metric is compact if and only if it is closed and bounded.

Corollary (Classical Bolzano-Weierstrass). Every bounded, infinite subset of $\mathbf{R}^n$ has a cluster point.

Connectedness

Definition. (1) A metric space (E; d) is <u>connected</u> if the only subsets of E that are both open and closed are E and ∅.

(2) A metric space (E; d) is <u>connected</u> if it cannot be written as a union of open sets A and B such that $A \cap B = \emptyset$, $A \cup B = E$, $A \neq \emptyset$, $B \neq \emptyset$.

Theorem. Let $\{S_i\}$ be a collection of connected subsets of a metric space. Suppose there exists i_0 such that $S_i \cap S_{i0} \neq \emptyset$. Then $\cup S_i$ is connected.

Theorem. A subset of **R** is connected if and only if it is an interval.

Theorem. $\mathbf{R}^n$ is connected.

Some other types of spaces

Definition. A <u>normed linear space</u> is a vector space (V) with a mapping $\|\ \|: V \to \mathbf{R}$ such that

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$.
3. $\|cx\| = c\|x\|$ where $c \in \mathbf{R}$.

4. $||x + y|| \le ||x|| + ||y||$.

Remark. A good candidate for the norm in a metric space is d(0, x).

Definition. An <u>inner product space</u> is a vector space, V, with a mapping $<,>: V \times V \rightarrow \mathbf{R}$, with the following properties:

1. $<x+y, z> = <x, z> + <y, z>$
2. $<cx, y> = c<x, y>$
3. $<x, y> = <y, x>$
4. $<x, x> \ge 0$
5. $<x, x> > 0$ if $x \ne 0$.

Definition. A complete inner product space is a <u>Hilbert space</u>.

Definition. A <u>Banach space</u> is a complete normed linear space.

Continuity

Definition. Let (E; d) and (E'; d') be metric spaces. Let f: E → E'. Let $p_0 \in E$. We say f is <u>continuous</u> at p_0 if given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $p \in E$ and $d(p, p_0) < \delta$ then $d'(f(p), f(p_0)) < \varepsilon$. We say f is continuous on E if f is continuous at all $p_0 \in E$. Alternately, f is continuous at p_0 if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B(p_0; \delta)) \subset B(f(p_0); \varepsilon)$.

Example. Let (E; d) be a metric space. Define $f: E \rightarrow \mathbf{R}$ by $f(p) = d(p, p_0)$, where p_0 is fixed. Then, f is continuous.

Example. A linear function, $f: \mathbf{R}^p \rightarrow \mathbf{R}^q$, is continuous.

Theorem. Suppose $f: \mathbf{R}^p \rightarrow \mathbf{R}^q$ is linear. Then there exists $A > 0$ such that, for all $u, v \in \mathbf{R}^p$, $|| f(u) - f(v) || \le A ||u - v||$.

Proposition. Let (E; d), (E'; d'), and (E''; d'') be metric spaces. Let f: E → E' and g: E' → E'' be continuous. Then g(f(x)) is continuous.

Definition. The pre-image of a set, S, is $f^{-1}(S) = \{p \in E \mid f(p) \in S\}$.

Theorem. Let (E; d) and (E'; d') be metric spaces, f: E → E'. Then, f is continuous if and only if for every open set $O \subset E'$, the pre-image of O is open.

Corollary. f is continuous if and only if the pre-image of every closed set is closed.

Note. A continuous function does not necessarily map open sets to open sets.

Proposition. Let E, E' be metric spaces. f: E → E' is continuous at p_0 if and only if for every sequence of points, $\{p_n\}$ that converges to p, $\{f(p_n)\}$ converges to f(p).

Theorem. Let (E; d) and (E'; d') be metric spaces and f: E→ E' be continuous. If $C \subset E$ is compact then $f(C) \subset E'$ is compact.

Corollary. A continuous real-valued function on a compact set attains a minimum and maximum value on at least one point on the set.

Theorem. Let (E; d) and (E'; d') be metric spaces and f: E → E' be continuous. If $C \subset E$ is connected then $f(C) \subset E'$ is connected.

Corollary (Intermediate Value Theorem). Let $f: \mathbf{R} \rightarrow \mathbf{R}$, $a, b \in \mathbf{R}$, $a < b$, f continuous on [a, b]. Then, for any γ strictly between f(a) and f(b), there exists $c \in (a, b)$ such that $f(c) = \gamma$.

Uniform Continuity

Definition. Let (E; d) and (E'; d') be metric spaces and f: E → E'. f is <u>uniformly continuous</u> if given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $p, q \in E$ with $d(p, q) < \delta$ then $d'(f(p), f(q)) < \varepsilon$.

Theorem. Let (E; d) and (E'; d') be metric spaces, and f: E → E' be continuous. If E is compact then f is uniformly continuous.

Sequences of Continuous Functions

Definition. Let (E; d) and (E'; d') be metric spaces and f_n: E → E' for all n. $\{f_n\}$ <u>converges</u> at p if $\{f_n(p)\}$ converges. $\{f_n\}$ <u>converges pointwise</u> on E if $\{f_n\}$ converges at p for all $p \in E$.

Definition. Let (E; d) and (E'; d') be metric spaces and f_n: E → E' for all n. The sequence $\{f_n\}$ <u>converges uniformly</u> to f if, given any $\varepsilon > 0$ there exists N such that, for all $p \in E$, if $n > N$, $d(f_n(p), f(p)) < \varepsilon$.

Theorem. Let (E; d) and (E'; d') be metric spaces. Let $\{f_n\}$ be a uniformly convergent sequence of continuous function. Then the limit of the sequence is continuous.

Definition. Let f_n: E → E'. $\{f_n\}$ is a <u>uniformly Cauchy sequence</u> if, given any $\varepsilon > 0$ there exists N such that, for all $n, m > N$ and for all $p \in E$, $d'(f_n(p), f_m(p)) < \varepsilon$.

Theorem. Let (E; d) and (E'; d') be metric spaces. Let E' be complete. Let f_n: E → E'. The sequence $\{f_n\}$ converges uniformly if and only $\{f_n\}$ is uniformly Cauchy.
Theorem. Let (E; d) and (E'; d') be metric spaces. Let E be compact and E' be complete. Let C(E) be the set of all continuous functions from E → E'. Let $d_c(f, g) = \max\{d'(f(p), g(p)) \mid p \in E\}$ be a function from $C(E) \times C(E) \to \mathbf{R}$. Then, $(C(E); d_c)$ is a complete metric space. Moreover, convergence in this metric space is equivalent to uniform convergence in E.
Corollary. The metric space $(C([0, 1]); L_\infty)$ is complete.

Contraction Mapping Principle
Definition. Let (E; d) be a metric space. Let f: E → E. f is Lipschitz if there exists A > 0 such that $d(f(x), f(y)) < A\, d(x, y)$ for all x, y ∈ E. f is contraction if A < 1.
Remark. Any Lipschitz function is uniformly continuous. (Choose $\delta < \varepsilon / A$.)
Theorem (Contraction Mapping Principle). Let E be a complete metric space. Let f: E → E be a contraction with contractive constant A. Then there exists a unique p such that f(p) = p. If $p_0 \in E$ then $\{f^n(p_0)\}$ converges to p, and $d(f^n(p_0), p) \le d(p, p_0)\, A^n / (1 - A)$.

Differentiation
Definition. Let f be a real-valued function on an open subset $U \subset \mathbf{R}$. Let $x_0 \in U$. f is differentiable at x_0 if $\lim_{x \to x_0} (f(x) - f(x_0))/(x - x_0)$ exists. The limit is denoted by $f'(x_0)$. Alternatively, f is differentiable if there exists $f'(x_0)$ such that, for any ε >0 there exists δ > 0 such that whenever $|x - x_0| < \delta$, $|f(x) - f(x_0) - f'(x_0)(x - x_0)| < \varepsilon |x - x_0|$.
Theorem. If f is differentiable at x_0 then f is continuous at x_0.
Proposition. Let f be a real-valued function on U then attains a maximum or minimum at $x_0 \in U$. Then, if f is differentiable, $f'(x_0) = 0$.
Rolle's Lemma. Let a, b ∈ **R**, a < b. Let f be continuous on [a, b] and differentiable on (a, b). Suppose f(a) = f(b) = 0. Then there exists c ∈ (a, b) such that f'(c) = 0.
Mean Value Theorem. Let a, b ∈ **R**, a < b. Let f be continuous on [a, b] and differentiable on (a, b). Then, there exists c ∈ (a, b) such that f'(c) = (f(b) – f(a)) / (b – a) [equivalently, f(b) – f(a) = f'(c)(b – a)].
Cauchy Mean Value Theorem. Let f and g be continuous on [a, b] and differentiable on (a, b). Then there exists c ∈ (a, b) such that f'(c) (g(b) – g(a)) = g'(c) (f(b) – f(a)).
Corollary. If f'(x) < 0 (> 0) for all x ∈(a, b) then f is decreasing (increasing).
Proposition. Let I = [a, b]. Let f be continuous on I and differentiable on (a, b). Let f(I) ⊆ I. Suppose $|f'(t)| < \alpha < 1$ for all t ∈ I. Then f(x) = x has a unique solution, x_0, given by $x_0 = \lim (n \to \infty) f^n(x)$ and x ∈ I.
Taylors's Theorem. Let $U \subset \mathbf{R}$ be an open interval. Let f: U → **R** be n+1 times differentiable. Then, for any a, b ∈ U, $f(b) = f(a) + f'(a)(b-a)/1! + f''(a)(b-a)^2/2! + \dots + f^{(n)}(a)(b-a)^n/n! + f^{n+1}(c)(b-a)^{n+1}/(n+1)!$, where c ∈ (a, b).
Sequences of Differentiable Functions
Theorem. Let $\{f_n\}$ be a sequence of real-values functions which are continuous on [a, b] and differentiable on (a, b). Let $\{f_n\}$ converge pointwise to f and $\{f_n'\}$ converge uniformly to ϕ. Then, $\{f_n\}$ converges uniformly to f, f is differentiable, and f' = ϕ.

Integration Theory
Definition. A partition of $[a, b] \subset \mathbf{R}$ is a finite sequence of numbers $x_0, x_1, \dots, x_n$, such that $a = x_0 < x_1 < \dots < x_n = b$. The width or (mesh) of this partition is $\max \{x_i - x_{i-1} \mid i = 1, \dots, n\}$.
Definition. The Riemann sum for f corresponding to *P* is $S(f; P) = \sum f(x_i')(x_i - x_{i-1})$ where $x_i' \in [x_{i-1}, x_i]$.
Definition. F is Riemann integrable on [a, b] if the sequence of partial Riemann sums for any partition *P* of [a, b] converges as the width of *P* approaches 0. Given ε > 0 there exists δ > 0 such that whenever width(*P*) < δ, $|S(f; P) - \int_a^b f(x)\,dx| < \varepsilon$.
Fact. Constant and step functions are integrable.

Definition. Let f be a bounded function on [a, b]. Let P be a partition of [a, b]. The <u>upper and lower Darboux sums</u> are $\overline{S}(f; P) = \sum M_i (x_i - x_{i-1})$ where $M_i = \sup \{f(x) \mid x_{i-1} \leq x \leq x_i\}$ and $\underline{S}(f; P) = \sum m_i (x_i - x_{i-1})$ where $m_i = \inf \{f(x) \mid x_{i-1} \leq x \leq x_i\}$.

Remark. If P is replaced by a finer partition, P', then $\underline{S}(P; f) \leq \underline{S}(P'; f) \leq \overline{S}(P'; f) \leq \overline{S}(P; f)$.

Definition. The upper and lower Darboux integrals are the limits of the Darboux sums as the width of the partitions approach 0.

Remark. Because the upper Darboux sums are decreasing and bounded from below (by the lower Darboux sums), they must converge. Similarly, the lower Darboux sums must converge.

Remark. $\underline{S}(P; f) \leq S(P; f) \leq \overline{S}(P; f)$ for any P.

Theorem. The Riemann integral on [a, b] exists if and only if the upper and lower Darboux integrals are equal.

Corollary. f is Riemann integrable on [a, b] $\Leftrightarrow$ for every ε there exists δ such that whenever width $P < \delta$, $0 \leq$ upper-S(f; P) – lower-S(f; P) $\leq \varepsilon$.

Theorem. Every continuous function on [a, b] is integrable.

Proposition. $\int f(x)\,dx + \int g(x)\,dx = \int f(x) + g(x)\,dx$ [over the same intervals].

Proposition. $\int c\, f(x)\,dx = c \int f(x)\,dx$ [over the same intervals].

Proposition. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$. (If any two of these exist, the third must exist.)

Proposition. If $f(x) \geq g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)$.

Fundamental Theorem of Calculus. Let f be a continuous function on an open interval $U \subset \mathbf{R}$. Let $a \in U$. Let $F(x) = \int_a^x f(t)\,dt$. Then, F is differentiable on U, and F' = f.

Corollary. If f is continuous, then f is the derivative of some function which may be defined by $F(x) = \int_a^x f(t)\,dt$.

Corollary. If F is a real-valued function on [a, b] that has derivative f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Theorem. Let $[a, b] \subset \mathbf{R}$. Let $\{f_n\}$ be a sequence of integrable functions on [a, b] that converges uniformly to f. Then, f is integrable and $\lim \int f_n(x)\,dx = \int f(x)\,dx$.

Series of Functions

Definition. Let $\{f_k\}$ be a sequence of functions from $(E; d) \rightarrow \mathbf{R}$. We define $S_n(x) = \sum f_k(x) = f_1(x) + \ldots f_n(x)$. If $\{S_n(x)\}$ converges to f(x) on E, then we say $\sum f_k(x) = f(x)$.

Definition. If $\sum |f_k(x)|$ converges, then $\sum f(x)$ converges absolutely. If $\{S_n(x)\}$ converges uniformly, then $\sum f_k(x)$ converges uniformly.

Theorem. Suppose f_k is continuous for all k. If $\sum f_k(x)$ converges uniformly, then $\sum f_k(x)$ is continuous.

Theorem. Suppose f_k is integrable for all k. If $\sum f_k(x)$ converges uniformly, then $\sum f_k(x)$ is integrable.

Theorem. Suppose f_k is differentiable, $\sum f_k$ converges pointwise to f, and $\sum f'$ converges uniformly to ϕ. The, $\sum f_k$ converges uniformly to f, and f is differentiable with derivative ϕ.

Proposition. Let $\{f_k\}$ be a sequence of functions. $\sum f_k$ is uniformly convergent $\Leftrightarrow$ for all $\varepsilon > 0$ there exists M such that, for all m, n > M, m > n, $|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_m(x)| < \varepsilon$.

Theorem (Weierstrass M Test). Let M_k be a sequence of non-negative numbers such that $|f_k(x)| < M_k$ for all k, x. If $\sum M_k$ converges then $\sum f_k$ is uniformly convergent.

Measure and Integrability [optional]

Definition. If I is a bounded interval with endpoints a and b we define the <u>length</u> of I by $|I| = |b - a|$.

Definition. A subset $A \subset \mathbf{R}$ is a set of measure 0 if, for all $\varepsilon > 0$ there exists a sequence of bounded intervals, $I_1, I_2, \ldots$, such that $A \subset \cup I_n$ and $\sum |I_n| < \varepsilon$.

Remark. No closed interval with $a \neq b$ has measure 0.

Remark. Any countable set of points has measure 0.

Remark. Any subset of a set of measure 0 has measure 0.

Remark. The union of a countable number of sets of measure 0 has measure 0.

Remark. There are uncountable sets of measure 0. Consider the Cantor set.

Definition. A property that holds except on a set of measure 0 is said to hold <u>almost everywhere.</u>

Definition. For a sequence of real numbers, $\{a_n\}$, we define the lim sup as $\limsup_{n\to\infty} = \lim_{N\to\infty}\sup \{a_n \mid n > N\}$.

Definition. The liminf is defined by $\liminf_{n\to\infty}\{a_n\} = \lim_{N\to\infty} \inf\{a_n \mid n > N\}$.

Remark. Because $\{a_n \mid n > N\} \subset \{a_n \mid n > N-1\}$, $\sup\{a_n \mid n > N\}$ is decreasing. Therefore, it converges if it is bounded below. Similarly for the liminf.

Remark. $\{a_n\}$ converges if and only if $\liminf \{a_n\} = \limsup \{a_n\}$ (and both exist).

Definition. Let $f: [a, b] \to \mathbf{R}$, $c \in [a, b]$. We define $\limsup_{x\to c} f(x) = \lim_{\delta\to 0} \sup\{f(x) | x\in [a, b]\cap[c-\delta, c+\delta]\}$. We define $\liminf_{x\to c} f(x) = \lim_{\delta\to 0} \inf\{f(x)|x\in [a, b]\cap[c-\delta,c+\delta]\}$.

Definition. The <u>oscillation</u> of f at c is $\omega(f; c) = \limsup_{x\to c} f(x) - \liminf_{x\to c} f(x)$.

Remark. If f is continuous ar c then $\omega(f; c) = 0$.

Proposition. If $\omega(f; c) < \varepsilon$ for all $c \in [a, b]$ then there exists $\delta > 0$ such that if $x, y \in [a, b]$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Theorem. A bounded function is Riemann integrable if and only if it is continuous almost everywhere.