Topology Summary

Background Information

Well-Ordering, Induction, and S_W

Definition. A set A is <u>well-ordered</u> if every non-empty subset of A has a smallest element.

Theorem (Well-Ordering). If A is a set then there exists an order relation on A that is a well-ordering.

Definition. Let X be a well-ordered set. Let $a \in X$. The $S_a = \{x \in X \mid x < a\}$ is called the <u>section</u> of X by a.

Lemma. There exists a well-ordered set A having a largest element, Ω, such that S_Ω of A is uncountable but every other section is countable.

Theorem. If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.

Defintion. Let J be a well-ordered set. A subset $J_0 \subset J$ is <u>inductive</u> if, for every $a \in J$, $S_a \subset J_0$ implies that $a \in J_0$.

Principle of Transfinite Induction. If J is a well-ordered set and J_0 is an inductive subset, then $J = J_0$.

Theorem. Let J and C be well-ordered. Assume that there is no surjective mapping of a section of J into C. Then there exists a unique function $h: J \to C$ such that $h(x) = \textit{smallest}[C - h(S_x)]$ for all $x \in J$.

Groups (Particularly Free and Free Abelian Ones)

Definition. Let G be an abelian group and $\{G_\alpha\}$ a family of subgroups of G. We say that the subgroups $\{G_\alpha\}$ <u>generate</u> G if each $x \in G$ can be written as the finite sum of elements from the G_α; that is, $x = \sum x_\alpha$, with all but finitely many $x_\alpha = 0$. In this case, we say G is the <u>sum</u> of the G_α.

Definition. Let G be a group and $\{G_\alpha\}$ a family of subgroups of G. We say the $\{G_\alpha\}$ <u>generate</u> G if each $x \in G$ can be written as the finite product of elements of the G_α; that is, $x = x_1 \ldots x_n$.

Note. In $x_1 \ldots x_n$, we may only combine consecutive elements from the same subgroup. The word resulting form this is a <u>reduced word</u>.

Definition. If the expression $x = \sum x_\alpha$ is unique for all x, G is the <u>direct sum</u> of $\{G_\alpha\}$, and we write $G = \oplus G_\alpha$.

Definition. If the reduced word for x is unique for all x, then G is the <u>free product</u> of $\{G_\alpha\}$ and we write $G = \Pi^* G_\alpha$.

Lemma. $G = \oplus G_\alpha \Leftrightarrow$ given any abelian group H and any family of homomorphisms $\{h_\alpha: G_\alpha \to H\}$, there exists a unique $h: G \to H$ that agrees with h_α on each G_α.

Lemma. $G = \Pi^* G_\alpha \Leftrightarrow$ given any group H and any family of homomorphisms $\{h_\alpha: G_\alpha \to H\}$, there exists a unique homomorphism $h: G \to H$ that agrees with each h_α on each G_α.

Definition. Let $\{G_\alpha\}$ be abelian groups. Suppose G is abelian, and that $\{i_\alpha: G_\alpha \to G\}$ is a family of monomorphisms, such that $G = \oplus\ i_\alpha(G_\alpha)$. Then G is the <u>external direct sum</u> of $\{G_\alpha\}$ relative to $\{i_\alpha\}$.

Definition. Let $\{G_\alpha\}$ be groups. Suppose G is a group and $\{i_\alpha: G_\alpha \to G\}$ is a family of monomorphisms, such that $G = \Pi^*\ i_\alpha(G_\alpha)$. Then we say G is the <u>external direct product</u> of the groups$\{G_\alpha\}$ relative to the monomorphisms $\{i_\alpha\}$.

Theorem. Given a family of abelian groups $\{G_\alpha\}$, there exists an abelian group G which is their external direct sum (consider the Cartesian product).

Theorem. Given a family of groups $\{G_\alpha\}$ there exists a group G which is their external direct product (consider all words of finite length with elements from the groups).

Theorem. Let $\{G_\alpha\}$ be abelian groups. Suppose G and G' are abelian groups which are external direct products of the $\{G_\alpha\}$ (relative to families of monomorphisms, $\{i_\alpha\}$ and $\{i_\alpha'\}$). Then there is a unique isomorphism, $\phi: G \to G'$, such that $\phi \circ i_\alpha = i_\alpha'$ for each α.

Theorem. Let $\{G_\alpha\}$ be groups. Suppose G and G' are groups which are the external free products of the $\{G_\alpha\}$ relative to monomorphisms $\{i_\alpha\}$ and $\{i_\alpha'\}$. Then there is a unique isomorphism, $\phi: G \to G'$, such that $\phi \circ i_\alpha = i_\alpha'$ for each α.

Definition. Let G be an abelian group and $\{a_\alpha\}$ a family of elements of G. Let G_α be the subgroup generated by a_α. If the $\{G_\alpha\}$ generate G, then we say the elements $\{a_\alpha\}$ generate G. If each G_α is infinite cyclic and G is the direct sum of the $\{G_\alpha\}$, then G is a free abelian group with $\{a_\alpha\}$ as a basis.

Definition. Let $\{a_\alpha\}$ be a family of elements of a group G. Suppose each a_α generates an infinite cyclic subgroup G_α of G. If G is the free product of the $\{G_\alpha\}$, then G is a free group with system of free generators $\{a_\alpha\}$.

Lemma. G is a free abelian group with basis $\{a_\alpha\}$ $\Leftrightarrow$ for any abelian group H and any family of elements $\{y_\alpha\} \subset H$, there is a unique homomorphism $h: G \to H$ with $h(a_\alpha) = y_\alpha$ for each α.

Lemma. G is a free group with system of free generators $\{a_\alpha\}$ $\Leftrightarrow$ for any group H and any family of elements $\{y_\alpha\} \subset H$, there is a unique homomorphism, $h: G \to H$ with $h(a_\alpha) = y_\alpha$.

Theorem. If G is free abelian, the size of the basis is uniquely determined by G and is called the rank of G.

Corollary. If G is a free group, the number of elements in the system of free generators is unique.

Definition. Let G be a group. If $x, y \in G$, we define $[x, y] = xyx^{-1}y^{-1}$ to be the commutator of x and y. The subgroup generated by all the commutators in G called the commutator subgroup, $[G, G]$.

Lemma. $[G, G]$ is a normal subgroup of G and the quotient group $G/[G, G]$ is abelian. If $h: G \to H$ is any homomorphism with H abelian, $[G, G] \subset \ker h$., so that h induces a homomorphism, $k: G/[G, G] \to H$.

Proposition. Let F be the free group generated by $a_1, \ldots, a_n$. Let $x \in F$, N the least normal subgroup containing x, and $G = F/N$. If $p: F \to F/[F, F]$ is the projection homomorphism, then $F/[F, F]$ is free abelian with basis $\{p(a_1), \ldots, p(a_n)\}$. The abelianization of G ($= G/[G, G]$) is isomorphic to $(F/[F,F])/N'$, where N is the subgroup generated by $p(x)$.

Definition. Let H_0 be a subgroup of G. The normalizer is $N(H_0) = \{a \in G \mid aH_0a^{-1} = H_0\}$. This is the largest subgroup of G in which H_0 is normal.

Definition. Let G be an abelian group. Then $G = \mathbf{Z}_a \oplus \ldots \oplus \mathbf{Z}_k \oplus \mathbf{Z}^b$. We define the Betti number of G to be b (the rank of the free part of G). If we have a sequence of homology groups, we define β_q to be the Betti number of H_q.

Point-Set Topology

Definition. A topology on a set X is a collection of subsets of X, *T*, such that (1) $\emptyset, X \in T$, (2) if $A_\alpha \in T$ for each α, then $\cup A_\alpha \in T$, and (3) if $A_1, \ldots, A_n \in T$, then $\cap A_i \in T$. We call the sets in *T* open sets.

Definition. A basis for a topology on X is a collection of subsets of X, *B*, such that (1) For all $x \in X$, there is some $B \in \mathcal{B}$ with $x \in B$, and (2) if $x \in B_1 \cap B_2$ there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. $U \subset X$ is open in the topology generated by *B* if, for all $x \in U$, there is some $B \in \mathcal{B}$, such that $x \in B \subset U$.

Definition. A subbasis, *S*, for a topology on X is a collection of subsets of X whose union is X. The topology generated by *S* is the topology with basis being the collection of all finite intersections of elements in *S*.

Lemma. Let X be a topological space. Let *C* be a collection of open sets of X such that, for all open sets U and all $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subset U$. Then *C* is a basis for the topology of X.

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are topologies on a set X. If $\mathcal{T}' \supset \mathcal{T}$ then $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is coarser than $\mathcal{T}'$. If this containment is strict, then $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ and $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$.

Lemma. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $x \in X$ and $B \in \mathcal{B}$ there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Definition. Let X be a simply ordered set with at least two elements. The order topology on X has basis $\{(a, b) \mid a < b\} \cup \{[a_0, b) \mid a_0$ is the smallest element of $X\} \cup \{(a, b_0] \mid b_0$ is the largest element of $X\}$.

Definition. Let X and Y be topological spaces. The product topology on X and Y is given by the basis $\mathcal{T}_X \times \mathcal{T}_Y$.

Definition. Let X be a topological space. Let $Y \subset X$. The subspace topology on Y is given by $\mathcal{T}_Y = \{U \cap Y \mid U \text{ is open in } X\}$.

Theorem. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for topologies on X and Y. Then, $\mathcal{B} \times \mathcal{C} = \{U \times V \mid U \in \mathcal{B} \text{ and } V \in \mathcal{C}\}$ is a basis for the product topology on X and Y.

Theorem. $\mathcal{B}_A = \{A \cap B \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on $A \subset X$.

Note. The product of subspace topologies is not the subspace of the product of the topologies.

Definition. In a topological space X, A is closed if X – A is open.

Theorem. Let A be a subset of a space X. Then $x \in$ A-closure $\Leftrightarrow$ every open set containing x intersects A.

Definition. Let X be a topological space. A sequence of points, $x_1, x_2, \ldots$, converges to $x \in X$ if, for each neighborhood, U, of x, there exists N such that $x_n \in U$ for all $n \geq N$.

Note. Sequences that converge in one topology may not converge in a finer topology. In some topologies, sequences may converge to more than one (or any!) point.

Definition. A space is Hausdorff if, for any $x_1, x_2 \in X$ there exist disjoint open sets, U_1 and U_2 with $x_1 \in U_1$ and $x_2 \in U_2$.

Definition. A space is T_1 if one-point sets are closed.

Note. Hausdorff $\Rightarrow T_1$.

Proposition. X is Hausdorff $\Leftrightarrow \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Continuous Functions

Definition. A function $f: X \to Y$ is continuous if $f^{-1}(V)$ is open in X for each V that is open in Y.

Note. Let X and X' be the same space with different topologies. The identity function $i: X \to X'$ is continuous $\Leftrightarrow$ the topology of X is finer than the topology of X'.

Theorem. Let $f: X \to Y$ be a function. The following are equivalent:

- f is continuous
- For all $A \subset X$, f(A-closure) $\subset$ f(A)-closure
- $f^{-1}(C)$ is closed if C is closed
- For every $x \in X$ and every neighborhood, V, of f(x), there is a neighborhood U of x such that $f(U) \subset V$.

Definition. Suppose $f: X \to Y$ is a bijection. If both f and f^{-1} are continuous, we say f is a homeomorphism.

Theorem. Composites of continuous functions are continuous.

Theorem. $f: X \to Y$ is continuous if we can write X as the union of open sets, $\{U_\alpha\}$, such that $f \mid U_\alpha$ is continuous for each α.

Theorem (Pasting Lemma). If $f \mid A$ and $g \mid B$ are continuous, A and B are closed, and $f(x) = g(x)$ on $A \cap B$, then there is a continuous function on $A \cup B$ that agrees with f on A and g on B.

Definition. Let J be an index set. Given a set X, we define a J-tuple of X to be a function $x: J \to X$. If $\alpha \in J$, we write x_α for $x(\alpha)$. We consider this as the α^{th} coordinate, writing $\mathbf{x} = (x_\alpha)_{\alpha \in J}$. Let $\{A_\alpha\}$ be a family of sets. Let $X = \cup A_\alpha$. Then, ΠA_α is the set of all J-tuples of X with $x_\alpha \in A_\alpha$ for all $\alpha \in J$. We define the projection function $\pi_\beta(\mathbf{x})$ to be the β^{th} coordinate of $\mathbf{x}$.

Definition. Let ΠX_α be the product of a family of topologies. The box topology is the topology with basis $\{\Pi U_\alpha \mid U_\alpha$ is open in $X_\alpha\}$.
Definition. Let ΠX_α be the product of a family of topologies. The product topology is the topology with basis $\{\Pi U_\alpha \mid U_\alpha$ is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J\}$.
Note. If J is infinite, the box topology is strictly finer than the product topology.
Theorem. Let $f: A \rightarrow \Pi X_\alpha$ be given by $f(a) = (f_\alpha(a))$ where each $f_\alpha: A \rightarrow X_\alpha$ is continuous. In the product topology, f is continuous $\Leftrightarrow$ each f_α is continuous.
Theorem. Let $S = \cup_{\alpha\in J}\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha$ is open in $X_\alpha\}$. S is a subbasis for the product topology.
Note. The product topology is the coarsest topology such that the projection functions out of it are continuous.
Theorem. In the box and product topologies, the Cartesian product preserves subspace relationships, Hausdorff-ness, and closures (Π A-closure = (Π A)-closure.)
Definition. If d is a metric on a set X then the collection of all ε-balls, $B_d(x, \varepsilon)$ for all $x \in X$ and $\varepsilon > 0$ is a basis for the metric topology on X induced by d.
Definition. If X is a topological space, X is metrizable if there is a metric that induces the given topology on X.
Definition. Let X be a metric space with metric d. The standard bounded metric, d-bar, is d-bar(x, y) = min{d(x, y), 1}.
Definition. Given an index set J and points $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ of $\mathbf{R}^J$, we define a metric ρ-bar on $\mathbf{R}^J$ by ρ-bar($\mathbf{x}$, $\mathbf{y}$) = sup{d-bar(x_α, y_α)}. This is the uniform metric on $\mathbf{R}^J$ and induces the uniform topology.
Note. $B(\mathbf{x}, \varepsilon) = \cup_{\delta < \varepsilon} (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times \ldots$
Theorem. In $\mathbf{R}^J$, the uniform topology is finer than the product topology and coarser than the box topology.
Theorem. Let d-bar be the standard bounded metric on $\mathbf{R}$. If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^\omega$, define D($\mathbf{x}$, $\mathbf{y}$) = sup{d-bar(x_i, y_i)/i}. Then D induces the product topology.
Theorem. Let $f:X \rightarrow Y$. Let X and Y be metrizable with metrics d_x and d_y. f is continuous $\Leftrightarrow$ for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d_y(f(x), f(y)) < \varepsilon$ whenever $d_x(x, y) < \delta$.
Sequence Lemma. Let X be a topological space, $A \subset X$. If there is a sequence of point in A converging to x, then $x \in$ A-closure. If X is metrizable, then the converse holds.
Corollary. $\mathbf{R}^\omega$ in the box topology is not metrizable.
Definition. Let $f_n: X \rightarrow Y$ be a sequence of functions with Y a metric space with metric d. We say the sequence (f_n) converges uniformly to the function $f: X \rightarrow Y$ if, for all $\varepsilon > 0$, there exists N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and $x \in X$.
Note. Let $f_n: X \rightarrow \mathbf{R}$. (f_n) converges uniformly to $f: X \rightarrow \mathbf{R} \Leftrightarrow$ the sequence $(f_n) \in \mathbf{R}^X$ converges to f in the uniform metric.
Definition. Let $p: X \rightarrow Y$ be surjective. p is a quotient map if $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X.
Definition. Let X be a space and Y a set. Let $p: X \rightarrow Y$ be onto. Then the quotient topology on T, T_Y is given by $T_Y = \{V \mid p^{-1}(V)$ is open in X$\}$.
Theorem. The quotient topology on Y is the finest topology such that p is continuous.
Definition. Let $f: X \rightarrow Y$. If f(U) is open whenever U is open, then f is an open map. If f(C) is closed whenever C is closed, then f is a closed map.
Proposition. If $p: X \rightarrow Y$ is surjective, continuous, and either open or closed, then it is a quotient map.
Facts. Let $p: X \rightarrow Y$ and $q: X \rightarrow X^*$ be quotient maps.

- The typical open set of X^* is a collection of equivalence classes whose union is open in X.

- If A is a subspace of X, the restriction p_0: A → p(A) need not be a quotient map. If A is a saturated open (closed) set or p is open (closed), then p_0 is a quotient map.
- The composite of quotient maps is a quotient map.
- Any Cartesian product of quotient maps might not be a quotient map.
- X^* need not be Hausdorff, even if X is.
- If g: X → Z is constant on each $p^{-1}(\{y\})$, $y \in Y$, there exists f: Y → Z such that $g = f \circ p$. f is continuous or a quotient map if and only if g is.

Definition. A topological group is a group, G, which is also a topological space that satisfies the T_1 axiom, such that group multiplication and inversion are continuous maps.

Facts. Let G be a topological group.

- Let C be closed in G and $a \in G$. Then Ca, aC, and aC^{-1} are closed.
- Let U be open and S be any set in G. Then US, SU, and U^{-1} are open.
- G is regular.
- Let N be the component of the identity in G. Then N is a normal subgroup.
- If N is the component of the identity, then G/N is totally disconnected.

Connectedness

Definition. Let X be a topological space. A separation of X is a pair of subsets of X that are disjoint, non-empty, and open, whose union is X.

Definition. If there is no separation of X, then X is connected.

Theorem. Suppose $A \subset X$ is connected and $A \subset B \subset$ A-closure. Then B is connected.

Theorem. The image of a connected space under a continuous map is connected.

Theorem. A finite product of connected spaces is connected.

Theorem. An infinite product of connected spaces is connected in the product topology, but not in the uniform or box topology.

Definition. A simply ordered set, L, with more than one element is a linear continuum if (1) L has the least upper bound property, and (2) if $x < y$, there exists $z \in L$ such that $x < z < y$.

Theorem. If L is a linear continuum with the order topology, then L and any intervals and rays in L are connected.

Definition. Let $x, y \in X$. A path from x to y is a continuous map f: [a, b] → X such that f(a) = x and f(b) = y. A space X is path-connected if there is a path between every pair of points in X.

Note. Path connected implies connected, but not vice versa.

Theorem. $S_\Omega \times [0, 1)$ is well-ordered. Remove the smallest element, a_0. Then the remainder, L, is the "long line". L is path-connected and locally homeomorphic to **R**, but cannot be imbedded in any $\mathbf{R}^n$.

Definition. Given a topological space X, we may define an equivalence relation on X, where x ~ y if there is some connected subspace of X containing both x and y. We call the equivalence classes of this relation the components of X.

Theorem. The components of X are connected disjoint subsets of X whose union is X, such that each non-empty connected subspace of X intersects only one subset.

Definition. A path component is defined by the equivalence relations where x ~ y if x and y are connected by a path.

Fact. Components are always closed in X. Components are open if there are finitely many components (**Q** is a counterexample).

Definition. X is locally connected at $x \in X$ if, for each neighborhood U of x, there is a connected neighborhood of x contained in U. X is locally path connected at x if each neighborhood of x contains a path connected neighborhood of x.

Note. Locally path connected implies locally connected.

Theorem. X is locally connected ⇔ for each open set U in X, each component of U is open in X.

Theorem. X is locally path connected ⇔ for each open U in X, each path component of U is open in X.

Note. If X has a basis of connected sets, it is locally connected.
Theorem. Each path component of X lies in a component of X. X is locally path connected if and only if the components and the path components are the same.
Definition. X is <u>weakly locally connected</u> at x if, for every open neighborhood UU of x, there exists a connected subspace of x that is contained in U that contains a neighborhood of x.
Proposition. If X is weakly locally connected at each point, then X is locally connected.
Compactness
Definition. Let X be a topological space. Let *A* be a collection of subsets of X. *A* <u>covers</u> X if the union of all the sets in *A* is X. *A* is an <u>open cover</u> if each element of *A* is open in X.
Definition. A space X is compact if every open covering contains a finite subcovering.
Lemma. Let Y be a subspace of X. Then Y is compact ⇔ every open covering of Y be sets open in X contains a finite subcover of Y.
Theorem. Every compact subspace of a Hausdorff space is closed.
Theorem. The image of a compact set under a continuous map is closed.
Theorem. A subspace $A \subset \mathbf{R}^n$ is compact ⇔ A is closed and bound in the Euclidean (or square) metric.
Theorem (Extreme Value). Let f: X → Y be continuous. Let Y be an ordered set. If X is compact, then there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.
Tube Lemma. Consider $X \times Y$ with Y compact. If $N \subset X \times Y$ is open and contains a slice $x_0 \times Y$ then N contains some tube $W \times Y$ where W is an open neighborhood of x_0 in X.
Note. If G is a topological space, with A closed and B compact, then AB is closed.
Definition. A collection, *C*, of subsets of X has the <u>finite intersection property</u> if, for every finite subcollection, $\{C_1, \ldots, C_n\}$, their intersection is non-empty.
Theorem. X is compact ⇔ for every collection, *C*, of closed sets in X with the finite intersection property, the intersection of these sets is non-empty.
Corollary. A collection of nested sets in a compact space has a point in common.
Definition. A space X is <u>limit point compact</u> if every infinite subset of X has a limit point in X.
Theorem. Every compact space is limit point compact.
Definition. Let (x_n) be a sequence of points in X. Let $n_1 < n_2 < \ldots$ be an infinite sequence of increasing integers. Then the sequence (x_{ni}) is a <u>subsequence</u>.
Definition. A space X is <u>sequentially compact</u> if every sequence has a convergent subsequence.
Note. Sequential compactness is also weaker than compactness.
Theorem. If X is metrizable, then compactness, limit point compactness, and sequential compactness are equivalent.
Definition. X is <u>locally compact at x</u> $\in$ X if there is some compact subspace that contains a neighborhood of x.
Note. Compact implies locally compact.
Theorem. X is locally compact Hausdorff ⇔ there exists Y such that (1) X is a subspace of Y, (2) Y – X consists of a single point, and (3) Y is compact. Any two such Y are homeomorphic, with the homeomorphism equal to the identity on X.
Definition. Such a Y is the <u>one-point compactification</u> of X.
Note. To construct Y, we add a point, ∞, such that set are open if they are open in X or they are the complement of a compact set in X.
Countability Axioms
Definition. A <u>countable basis</u> at a point is a countable subset of basis elements such that any neighborhood of that point contains one of these basis elements.
Definition. X is <u>first-countable</u> if there is a countable basis at each $x \in X$.
Definition. X is <u>second-countable</u> if the topology of X has a countable basis.
definition. X is <u>Lindelof</u> of every open cover contains a countable subcover.
Definition. A topology has a <u>countable dense subset</u>, A, is A is countable and A-closure = X.

Note. D is dense if every non-empty open set in X intersects D. a is a limit point of S if every non-empty open set about a intersects S.

Separability Axioms

Definition. Suppose one-point sets are closed in X. X is <u>regular</u> if, for each pair consisting of a point $x \in X$ and a closed set $B \subset X$, $x \notin B$, there exist disjoint open sets containing x and B respectively. X is <u>normal</u> if, for each pair of disjoint closed sets A and B, there exist disjoint open sets containing A and B.

Note. Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff.

Lemma. Let X be a topological space in which one-point sets are closed. X is regular $\Leftrightarrow$ for all $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that V-closure $\subset$ U. X is normal $\Leftrightarrow$ for all closed sets $A \subset X$ and open sets U containing A, there exists an open set V such that $A \subset V$ and V-closure $\subset$ U.

Example. Metrizable spaces are normal.

Theorem. Subspaces and products of Hausdorff/regular spaces are Hausdorff/regular.

Algebraic Topology

Definition. Let f, f': X → Y be continuous maps. f is <u>homotopic</u> of f' if there is a continuous map F: $X \times I$ → Y such that F(x, 0) = f(x) and F(x, 1) = f'(x).

Lemma. Homotopy is an equivalence relation.

Definition. Let f, f': [0, 1] → Y be continuous maps. f is <u>path homotopic</u> to f' is there is a continuous map F: $[0, 1] \times [0, 1]$ → Y such that F(x, 0) = f(x), F(x, 1) = f'(x), F(0, t) = f(0) = f'(0) and F(1, t) = f(1) = f'(1).

Definition. The composition of two paths, f*g, is given by f*g: [0, 1] → X, with f*g(t) = f(2t) or g(2(t-1/2)).

Definition. Let $x_0 \in X$. The set of path homotopy classes of loops based at x_0, under path composition, is called the <u>fundamental group</u> relative to the basepoint x_0, and is denoted by $\pi_1(X, x_0)$.

Definition. Given a pointed set (X, x_0), $\pi_0(X, x_0)$ is the pointed set of path components.

Proposition. Let f: (X, x_0) → (Y, y_0), g: (W, w_0) → (X, x_0), with $f \circ g = h$. Then $f_*: \pi_1(X, x_0)$ → $\pi(Y, y_0)$ and $g_*: \pi_1(W, w_0)$ → $\pi_1(X, x_0)$ are homomorphisms, as is $h_* = f_* \circ g_*$. The same is true with π_0 instead of π_1.

Definition. If X is path connected and $\pi_1(X, x_0) = \{[x_0]\}$, then X is <u>simply connected</u>. If $\pi_0(X, x_0) = \{X\}$, then X is <u>path connected</u>.

Definition. Let α be a path from x_0 to x_1. Define $\alpha^{\wedge}: \pi_1(X, x_0)$ → $\pi_1(X, x_1)$ by $\alpha^{\wedge}([f]) = [\alpha$-reverse]*[f]*[$\alpha$].

Definition. Let p: E → B be continuous and surjective. Let $U \subset B$ be open. U is <u>evenly covered</u> by p if $p^{-1}(U) = \cup\{V_\alpha\}$ where each V_α is open, homeomorphic to U, and disjoint from all other $V_{\alpha'}$.

Definition. If every $b \in B$ has a neighborhood that is evenly covered, then we call p a <u>covering map</u> and E a <u>covering space</u> of B.

Note. If p: E → B is a covering map and B is regular, so is E.

Theorem. Let p: E → B be continuous and onto, with U evenly covered. If U is connected, then the partition of $p^{-1}(U)$ into slices is unique.

Definition. Let p: E → B be any map. If f is a continuous map, f: X → B, a <u>lifting</u> of f is a map f~: X →Y E such that p °f~ = f.

Lemma. Let p: E → B be a covering map. Let $p(e_0) = b_0$. Then any path f: [0, 1] → B beginning at b_0 has a unique lifting to a path f~ beginning at e_0 in E.

Lemma. Let $p: E \rightarrow B$ be a covering map. Let $p(e_0) = b_0$. Let $F: I \times I \rightarrow B$ be continuous, with $F(0, 0) = b_0$. There is a unique lifting of F to the continuous map $F\sim: I \times I \rightarrow E$, such that $F\sim(0, 0) = e_0$. If F is a path homotopy, so is $F\sim$.

Theorem. Let $p: E \rightarrow B$ be a covering map, with $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1. Let $f\sim$ and $g\sim$ be their liftings to paths in E beginning at e_0. If f and g are path homotopic, the $f\sim$ and $g\sim$ end at the same point and are path homotopic.

Definition. Let $p: E \rightarrow B$ be a covering map. Let $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $f\sim$ be the lifting of f to a path in E beginning at e_0. Let $\phi([f]) = f\sim(1)$. Then $\phi: \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$ is a well-defined set map. ϕ is the <u>lifting correspondence</u> derived from a covering map p.

Theorem. If E is path connected, $\phi: \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$ is surjective. If E is simply connected, then ϕ is bijective.

Theorem. If $p: E \rightarrow B$ is a covering map with basepoints e_0 and b_0, let $F = p^{-1}(b_0)$. Then, $* \rightarrow \pi_1(F, e_0) \rightarrow \pi_1(E, e_0) \rightarrow \pi_1(B, b_0) \rightarrow \pi_0(F, e_0) \rightarrow \pi_0(E, e_0) \rightarrow \pi_0(B, b_0) \rightarrow *$ is a long exact sequence.

Note. If G is a topological group, $H \subseteq G$ a closed subgroup such that $p: G \rightarrow G/H$ is a covering map, the sequence above is a long exact sequence of groups.

Homotopy Lifting Lemma. Let $p: E \rightarrow B$ be a covering map with $p(e_0) = b_0$. Let $F: I \times I \rightarrow B$ be continuous with $F(0, 0) = b_0$. Then F can be uniquely lifted to $F\sim: I \times I \rightarrow E$ which is continuous and has $F\sim(0, 0) = e_0$. If F is a path homotopy, then so is $F\sim$.

Definition. If $A \subset X$, a <u>retraction</u> of X onto A is a continuous map $r: X \rightarrow A$ such that $r \mid A$ is the identity map. If such an r exists, we call A a <u>retract</u> of X.

Lemma. If A is a retract of X then the homomorphism of fundamental groups induced by inclusion, $j: A \rightarrow X$, is injective.

Theorem. There is no retraction of B^2 onto S^1.

Lemma. Let $h: S^1 \rightarrow X$ be continuous. The following are equivalent:

- h is nulhomotopic
- h extends to a continuous map, $k: B^2 \rightarrow X$.
- h_* is the trivial homomorphism of fundamental groups.

Corollary. The inclusion map, $j: S^1 \rightarrow \mathbf{R}^2 - \mathbf{0}$ is not nulhomotopic. Neither is the identity map, $i: S^1 \rightarrow S^1$.

Definition. A <u>vector field</u> on B^{n+1} is an ordered pair $(\mathbf{x}, v(\mathbf{x}))$ where $\mathbf{x} \in B^{n+1}$, $v: B^{n+1} \rightarrow \mathbf{R}^{n+1}$ is continuous. We call a vector field <u>non-vanishing</u> if $v(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x}$. (So $v: B^{n+1} \rightarrow \mathbf{R}^{n+1} - \mathbf{0}$.)

Theorem. Given a non-vanishing vector field on B^2, there exists a point on S^1 where the vector field points directly outward and a point on S^1 where it points directly outward.

Theorem (Brouwer Fixed Point). If $f: B^2 \rightarrow B^2$ is continuous, then there is a point $x \in B^2$ with $f(x) = x$.

Corollary. Let A be a 3 by 3 matrix of positive real numbers. Then A has a positive real eigenvalue.

Theorem (Fundamental Theorem of Algebra). A polynomial equation, $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ of degree $n > 0$ with real or complex coefficients has at least one real or complex root.

Proof. Scale the equations so that any root must be in B^2. If there is no root, this defines a non-vanishing vector field, which must be nulhomotopic on S^1. Consider the map $f(z) = z^n$ on S^1. These maps are homotopic, but one is nulhomotopic and the other isn't.

Theorem. If $h: S^1 \rightarrow S^1$ is continuous and preserves antipodes, then h is not nulhomotopic.

Theorem (Borsuk-Ulam). Given a continuous map, $f: S^2 \rightarrow \mathbf{R}^2$, there exists $\mathbf{x} \in S^2$ with $f(\mathbf{x}) = f(-\mathbf{x})$.

Definition. Let A be a subspace of X. A is a <u>deformation retract</u> of X if the identity map of X is homotopic to the retraction of X onto A, such that each point of A remains fixed during the homotopy.

Definition. Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be continuous. Suppose $g \circ f : X \rightarrow X$ is homotopic to the identity map on X and $f \circ g: Y \rightarrow Y$ is homotopic to the identity map on Y. Then f and g are homotopy equivalences, and each is the homotopy inverse of the other. Then X and Y are of the same homotopy type.

Theorem. Suppose $X = U \cup V$, where U and V are open in X. Suppose $U \cap V$ is path connected and $x_0 \in U \cap V$. Let i, j be the inclusion maps of U and V into X. Then the induced homomorphisms of fundamental groups, $i_*: \pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$ and $j_*: \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$, generate $\pi_1(X, x_0)$.

Definition. A surface is a Hausdorff space with a countable basis, so that each point has a neighborhood homeomorphic to an open subset of $\mathbf{R}^2$.

Theorem. $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Definition. The projective plane, P^2, is the quotient space obtained by identifying each point, $\mathbf{x}$, of S^2 with $-\mathbf{x}$.

Definition. T#T, the double torus, is obtained by cutting circular holes in two tori and attaching them along the holes.

Theorem. Let A_1 and A_2 be closed connected subsets of S^2. Let $A_1 \cap A_2 = \{a, b\}$. Let $C = A_1 \cup A_2$. Then C separates S^2.

Corollary. Any two closed sets in S^2 that intersect in exactly two points must miss at least two points of S^2.

Theorem (Jordan Curve). Let C be a simple closed curve in S^2. Then C separates the sphere into exactly two components, of which C is the common boundary.

Theorem (Schoenflies). If the simple closed curve C separates S^2 into two components U and V, then U-closure and V-closure are homeomorphic to B^2. Let h be a homeomorphism of C with the equator of S^2. The h extends to a homomorphism $k: S^2 \rightarrow S^2$.

Definition. Theta space is the union of three arcs, all of which intersect at the two common endpoints.

Lemma. Let X be a theta space contained in S^2. Let A, B, C be arcs with $A \cup B \cup C = X$. Then X separates S^2 into exactly three components, with boundaries $A \cup B$, $A \cup C$, and $B \cup C$.

Theorem. $K_{3,3}$ cannot be imbedded in the plane.

Definition. Let $h: S^1 \rightarrow \mathbf{R}^2 - \mathbf{0}$ be continuous. Then $h_*: \pi_1(S^1) \rightarrow \pi_1(\mathbf{R}^2 - \mathbf{0})$ maps the generator of S^1 to some element, d, of $\mathbf{Z}$. d is the winding number of h.

Lemma. Let $h: S^1 \rightarrow \mathbf{R}^2 - \mathbf{0}$ with C = Im h, so that h is a homeomorphism of S^1 with a simple closed curve. Then, if $\mathbf{0}$ is in the unbounded component, the winding number is 0, and if $\mathbf{0}$ is in the bounded component, the winding number is +1 or –1.

Lemma. The inclusion map, $j: C \rightarrow S^2 - p - q$ induces an isomorphism of fundamental groups.

Definition. Let f be a loop in $\mathbf{R}^2$ with $a \in \mathbf{R}^2 - \text{Im } f$. Set $g(s) = (f(s) - a)/\|f(s) - a\|$. Then g is a loop in S^1. Let g~ be a lifting of g. The winding number of g is g~(1) – g~(0) = n(f, a).

Definition. Let $F: I \times I$ be continuous, with $F(0, t) = F(1, t)$, so that $f_t(s) = F(s, t)$ is a loop for all t. Then F is a free homotopy of loops.

Lemma. Let f be a loop in $\mathbf{R}^2 - a$. If f is freely homotopic to f' through loops in $\mathbf{R}^2 - a$, then n(f, a) = n(f', a).

Lemma. Let α be a path from a to b in $\mathbf{R}^2 - f(I)$. Then n(f, a) = n(f, b).

Fact. $n(f, a) = (\int_f dz/(z - a))/2\pi i$.

Theorem (Siefert-Van Kampen). Let $X = U \cup V$, where U and V are open in X. Let U, V, and $U \cap V$ be path connected. Let $x_0 \in U \cap V$. Let H be any group, with homomorphisms $\phi_1: \pi_1(U, x_0) \rightarrow H$ and $\phi_2: \pi_1(V, x_0) \rightarrow H$. Let $i_1: \pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$, $i_2: \pi_1(U \cap V, x_0) \rightarrow \pi_1(V, x_0)$, $j_1: \pi_1(U, x_0) \rightarrow \pi_1(U \cup V, x_0)$, and $j_2: \pi_1(V, x_0) \rightarrow \pi_1(U \cup V, x_0)$ be the homomorphisms induced by inclusion. If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there exists a unique homomorphism, $\Phi: \pi_1(X, x_0) \rightarrow H$, such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.

Theorem (Classical Seifert-Van Kampen). Let j: $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ extend the homomorphisms induced by inclusion, $j_1: \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_2: \pi_1(V, x_0) \to \pi_1(X, x_0)$. Then j is surjective and its kernel is the least normal subgroup of $\pi_1(U, x_0) * \pi_1(V, x_0)$ that contains $\{i_1^{-1}(g)i_2(g) \mid g \in \pi_1(U \cap V, x_0)\}$.

Corollary. If $U \cap V$ is simply connected, k: $\pi_1(U, x_0) * \pi_1(v, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

Corollary. If V is simply connected, $\pi_1(V, v_0) = \{e\}$, and there is an isomorphism k: $\pi_1(U, x_0)/N \to \pi_1(X, x_0)$, where N is the least normal subgroup of $\pi_1(U, x_0)$ containing the image of $i_1: \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$.

Definition. Suppose X is a space that is the union of closed subspaces, $X_1, \ldots, X_n$, with a single point $\{p\} = \cap X_i$. Then X is called the wedge of the spaces $X_1, \ldots, X_n$; we write $X = X_1 \vee \ldots \vee X_n$.

Theorem. Suppose that, for each i, that p is the deformation retract of an open set $W_i \subset X_i$. Then, $\pi_1(X, p)$ is the external free product of $\{\pi_1(X_i, p)\}$ relative to the monomorphisms induced by inclusion.

Definition. Let $X = \cup X_\alpha$. The topology of X is coherent with the subspaces X_α when a subset $C \subset X$ is closed if $C \cap X_\alpha$ is closed in X_α for each α.

Theorem. Let X be a wedge of the circles S_α; let p be the common point of these circles. Then $\pi_1(X, p)$ is a free group with system of free generators $\{f_\alpha \mid f_\alpha \text{ generates } \pi_1(S_\alpha, p)\}$.

Definition. Let n > 1. Let r: $S^1 \to S^1$ be rotation by the angle $2\pi/n$. Form a quotient space from B^2 by identifying the points of S^1 that are images of each other under rotation. This is the n-fold dunce cap.

Theorem. The fundamental group of the n-fold dunce cap is $\mathbf{Z}_n$.

Definition. Let X and Y be disjoint normal spaces, A closed in X, and f: A $\to$ Y be continuous. The adjunction space, Z_f, is the quotient space obtained from $X \cup Y$ by identifying $a \in A$ with f(a) and every point of $f^{-1}(\{f(a)\})$.

Theorem. Let X be a Hausdorff space. Let A' be a closed, path-connected subspace of X. Suppose h: $B^2 \to X$ is continuous, maps Int B^2 onto X – A' and S^1 = Bd B^2 into A'. Let $q \in S^1$ and a = h(q). Let k: $(S^1, q) \to (A', a)$ be the restriction of h. then, the homomorphism $i_*: \pi_1(A', a) \to \pi_1(X, a)$ induced by inclusion is surjective with kernel the least normal subgroup of $\pi_1(A', a)$ that contains Im k_*.

Note. In the theorem above, the space obtained by adding B^2 is the adjunction space of B^2 and X.

Note. We may use this to show that any group is the fundamental group of some space.

Definition. For a path-connected space, its first homology group is $H_1(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Definition. Let P be a polygonal region. Given orientations and labels of edges of P, we may completely describe this labelling by $w = a_1^{\pm1}\ldots a_k^{\pm1}$, where a_i is the label on the i^{th} edge (it is possible that $a_i = a_j$) and the exponent is +1 if the orientation is counterclockwise and –1 otherwise, This is called a labelling scheme of length n for P.

Note. A surface is obtained from a polygonal region by making a quotient space that identifies edges with the same labels, so that their orientations match.

Note. The following operations on polygonal schemes will not change the surface:

- Cut: Replace y_0y_1 by y_0c, $c^{-1}y_1$.
- Paste: Replace y_0c, $c^{-1}y_1$ by y_0y_1 if c is not used elsewhere in the scheme.
- Relabel: Replace all occurrences of one label by another (not-yet-used!) label, or change the signs on all the exponents of one label.
- Permute: Replace y_0y_1 by y_1y_0.
- Flip: Write the entire scheme backwards.
- Cancel: Replace $y_0cc^{-1}y_1$ by y_0y_1.

Definition. Let w be a proper labelling scheme. w is of torus type if each label in w appears once with exponent +1 and once with exponent –1. Otherwise, w is of projective type.

Theorem (Classification of Surfaces). Let X be the quotient space obtained from a polygonal region by pasting its edges together in pairs. Then X is homeomorphic to one of the following:

- Sphere: S^2 – $aa^{-1}bb^{-1}$
- Projective Plane: P^2 – abab
- M-fold Connected Sum of Projective Planes: P_m – $(a_1a_1)\ldots(a_ma_m)$, m ≥ 2
- N-fold Connected Sum of Tori: T_n – $(a_1b_1a_1^{-1}b_1^{-1})\ldots(a_nb_na_n^{-1}b_n^{-1})$, n ≥ 1

Covering Spaces

Definition. Let p: E → B and p': E' → B be covering maps. They are equivalent if there exists a homeomorphism h: E → E' such that p = p' ° h. h is called an equivalence of covering spaces.

Theorem. Let p: E → B and p': E' → B' be covering maps, with $p(e_0) = p'(e_o') = b_0$. There is an equivalence h: E → E' such that $h(e_0) = e_0'$ ⇔ $p_*(\pi_1(E, e_0)) = p_*'(\pi_1(E', e_0'))$.

Theorem. Let p: E → B and p': E' → B' be covering maps, with $p(e_0) = p'(e_o') = b_0$. There is an equivalence h: E → E' ⇔ $p_*(\pi_1(E, e_0))$ and $p_*'(\pi_1(E', e_0'))$ are conjugate in $\pi_1(B, b_0)$.

Definition. A covering space is a universal covering space if it is simply connected.

Theorem. If E and E' are universal covering maps of B, the E and E' are equivalent.

Lemma. Let p, q, and r be continuous maps, with p = r ° q. If p and r are covering maps, so is q. If p and q are covering maps, so is r. If q and r are covering maps and Im p = Im r has a universal covering space, p is a covering map.

Theorem. Suppose E is a universal covering space of B and Y is any other covering space. Then E is a covering space of Y.

Lemma. Let p: E → B be a covering map with $p(e_0)$ b_0. If E is simply connected, then b_0 has a neighborhood U such that the homomorphism induced by inclusion, $i_*: \pi_1(U, b_0) \to \pi_1(B, b_0)$ is trivial.

Definition. Let p: E → B be a covering map. We call the group of equivalences of E with itself the group of covering transformations, *C*(E, p, B).

Theorem. $N(H_0) / H_0$ is isomorphic to *C*(E, p, B).

Corollary. If H_0 is normal, $N(H_0) = \pi_1(B, b_0)$ and $C(E, p, B) = \pi_1(B, b_0)/H_0$.

Definition. When $p_*(\pi_1(E, e_0))$ is normal in $\pi_1(B, b_0)$, p is a regular covering map.

Theorem. If p: X → B is a regular covering map and G = *C*(X, p, B), then there is a homeomorphism k: X/G → B, such that p = k ° π, where π: X → X/G is projection.

Note. X/G is the quotient space obtained from identifying the orbit of a point under the covering transformations to a single point.

Definition. A space B is semilocally simply connected if each b ∈ B has a neighborhood U such that the homomorphism $i_*: \pi_1(U, b) \to \pi_1(B, b)$ induced by inclusion is trivial.

Theorem. Let B be path connected, locally path connected, and semilocally simply connected. Let $b_0 \in B$. Given a subgroup H of $\pi_1(B, b_0)$ there exists a space E and a covering map p: E → B with $e_0 \in p^{-1}(\{b_0\})$ such that $p_*(\pi_1(E, e_0)) = H$.

Proof (Construction). Let *P* be the set of all paths in B beginning at b_0. Define an equivalence relation on *P* by α ~ β if α(1) = β(1) and [α * β-reverse] ∈ H. Let E be the set of all equivalence classes, α#. Define p: E → B by p(α#) = α(1). Topologize E with the basis B(U, α) = {(α * δ)# | δ is a path in U beginning at α(1)}.

Exact Sequences, Chain Complexes, and Homology Groups

Definition. A pointed set (set with base point) is a pair (S, s_0). If G is a group, we assume (G, e) is the pointed set. If f: $(S, s_0) \to (T, t_0)$, we mean f: S → T and $f(s_0) = t_0$. Then, ker f = {s | f(s) $= t_0$}.

Definition. Let $\{A_i\}$ be a countable sequence of groups and $\{d_i: A_i \to A_{i-1}\}$ be homomorphisms. The sequence is a <u>chain complex</u> if each A_i is abelian and $d_i \circ d_{i+1}$ is trivial (that is, Im $d_{i+1} \subset$ Ker d_i).
Definition. Let $\{A_i\}$ be a sequence of groups and $\{d_i: A_i \to A_{i-1}\}$ be homomorphisms. The sequence is <u>exact at A_k</u> if Im d_{k+1} = Ker d_k. It is <u>exact</u> if it is exact at each A_k.
Definition. Let $C = (\{C_i\}; \{d_i\})$ be a chain complex. The <u>i^{th} homology group</u> of C is $H_i(C)$ = Ker d_i / Im d_{i+1}.
Corollary. C is exact $\Leftrightarrow$ $H_i(C)$ is trivial for all i.
Definition. Let $(\{C_i\}; \{c_i\})$ and $(\{D_i\}; \{d_i\})$ be chain complexes. A <u>chain map</u>, $\phi = \{\phi_i: C_i \to D_i\}$ is a collection of homomorphisms such that $\phi_i \circ c_{i+1} = d_{i+1} \circ \phi_{i+1}$.
Definition. If $\phi: C \to D$ is a map of chain complexes, $\phi_{i*}[x] = [\phi_i(x)]$ defines a map $\phi_{i*}: H_i(C) \to H_i(D)$.
Definition. Let $B = (\{B_i\}; \{b_i\})$ and $C = (\{C_i\}; \{c_i\})$ be chain complexes. Let f, g: $B \to C$ be chain maps. If there exists $D_n: B_n \to C_{n+1}$ for all n, such that $(c_{n+1}D_n + D_{n-1}b_n)(\sigma) = f(\sigma) - g(\sigma)$ for all $\sigma \in B_n$, then $\{D_n\}$ is a <u>chain homotopy</u> from f to g, and f and g are <u>chain homotopic</u>.
Theorem. If f, g: $B \to C$ are chain homotopic, then $H_nf = H_ng$ for every n. (That is, $(f_*([\sigma]) = g_*([\sigma])$.)
Definition. If f: $B \to C$ and g: $C \to B$ are chain maps, f and g are <u>chain homotopy inverses</u> of each other if $f \circ g: C \to C$ and $g \circ f: B \to B$ are chain homotopic to their respective identity maps.
Corollary. If f and g are chain homotopy inverses, then $H_nf = f_*$ and $H_ng = g_*$ are inverse homomorphisms and $H_n(C) = H_n(D)$.
Lemma (Zig-Zag or Fundamental Lemma of Homological Algebra).. Suppose we have chain complexes C, $D = (\{D_i\}, \partial)$, E, and chain maps f: $C \to D$ and g: $D \to E$, such that $0 \to C_n \to D_n \to E_n \to 0$ is exact for all n. We define $\partial_*: H_nE \to H_{n-1}C$ by $\partial_*[x] = [y]$ where $x \in Z_n(E)$, $y \in C_{n-1}$ and there exists $d \in D_n$ with $g_n(d) = x$ and $f_{n-1}(y) = \partial(d)$. Then ∂_* is a well-defined homomorphism, and the following sequence is exact: $\ldots \to H_nE \to H_{n-1}C \to H_{n-1}D \to H_{n-1}E \to \ldots$
Lemma. Let $C = (\{C_n\}; \{\partial_{Cn}\})$ and $D = (\{D_n\}; \{\partial_{Dn}\})$ be chain complexes. Then, $C \oplus D = (\{C_n \oplus D_n\}, \{(\partial_{Cn}, \partial_{Dn})\})$ is a chain complex, and $H_n(C \oplus D) = H_n(C) \oplus H_n(D)$.
Lemma. β_q is the dimension of $H_q(K, \mathbf{Q})$ as a vector space over $\mathbf{Q}$.
Simplicial Homology
Definition. Let $S = \{s_0, \ldots, s_n\}$ be a set of vertices (none of which lie in the hyperplanes spanned by the others). The n-<u>simplex</u> with vertex set S is $\sigma(S) = \{\sum t_is_i \mid \sum t_i = 1, t_i > 0\}$.
Definition. A <u>simplicial complex</u>, K, is a collection of simplices, such that (1) every face of a simplex in the collection is also in the collection, (2) every two simplices intersect in one common face (which may be $\emptyset$), and (3) the topology on the union is coherent with the topologies of the simplices. We define K_n to be the set of n-simplices in K.
Definition. Since, each simplex is a subspace of $\mathbf{R}^n$ for some n, the union of all the simplices is a subspace |K| of $\mathbf{R}^\infty$. |K| is called the <u>geometric realization</u> of K.
Definition. The unique simplex that a point lies in the interior of (ie., for which all the $t_i \neq 0$) is called the <u>carrier</u> of x, car(x).
Definition. If $A = \sigma(S)$ is a simplex, $S = \{v_0, \ldots, v_k\}$, the <u>barycenter</u> of A is $A^\wedge = (v_0 + \ldots + v_k) / (k+1)$.
Definition. If $A_0 < \ldots < A_k$ are nested simplices in a simplicial complex K, we write $\sigma(A_0, \ldots, A_k)$ for the simplex with vertices at the barycenters of each A_j, which is $\sigma(A_0^\wedge, \ldots, A_k^\wedge)$. The collection of all such simplices is called the <u>barycentric subdivision</u> of K, or K^1.
Definition. If K is a simplicial complex, the <u>mesh</u>, $\mu(K)$, is the maximum of the diameters of the simplices in the complex.

Lemma. If K is a simplicial complex such that the dimension of K and K^1 are both n, the $\mu(K^1) \leq \mu(K)\, n/(n+1)$.

Definition. A simplicial map, f: K → L, takes simplices of K linearly onto simplices of L. Equivalently, $f_0: K_0 \to L_0$ and whenever $\sigma(S) \in K$, $f(\sigma(S)) = \sigma(f(S)) \in L$.

Definition. A simplicial map, f: K → L, determines a simplicial map, |f|: |K| → |L| by $|f|(\sum t_i s_i) = \sum t_i f(s_i)$.

Definition. If f: |K| → |L| is a continuous map, a simplicial approximation is a simplicial map, s: K → L, such that $|s|(x) \in car(f(x))$ for all $x \in |K|$.

Note. Vertices are their own carriers, so |s|(u) = f(u) when f(u) is a vertex in |L|.

Theorem (Simplicial Approximation). If K is finite and if f: |K| → |L| is continuous, then for sufficiently large n, there is a simplicial approximation, s: $K^n \to L$ to f: $|K^n| \to |L|$.

Note. A simplicial approximation is always homotopic to the original function.

Definition. We define $C_n(K)$ to be the set of finite formal linear combinations of K_n with coefficients in **Z**.

Definition. Let $v_0, \ldots, v_n$ be an ordered set of vertices of a simplex. We define $\partial(v_0, \ldots, v_n) = \sum(-1)^i(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ and $\partial(v_0) = 0$.

Note. $\partial(\partial(v_0, \ldots, v_n)) = 0$ for all n. So $(\{C_n(K)\}, \{\partial: C_n(K) \to C_{n-1}(K)\})$ is a chain complex.

Definition. Suppose K is a simplicial complex and v is not in the hyperplane spanned by any finite subset of K_0.For each $S = \sigma(v_0, \ldots, v_n) \in K$, let $D_nS = \sigma(v, v_0, \ldots, v_n)$. The cone on K, CK, with vertex v is the simplicial complex which contains K and D_nS for each $S \in K$.

Theorem. Let K be any simplicial complex. Let v_0 be the cone vertex of CK. Let f, g: CK → CK be simplicial maps (which induce chain maps). Let f be the identity map and $g(v) = v_0$ for all $v \in CK_0$. Then f is chain homotopic to g.

Theorem. If K is a simplicial complex, then $H_0(CK) = \mathbf{Z}$ and $H_n(CK) = 0$ otherwise.

Corollary. If K is a k-simplex together with all of its faces, $H_0(K) = \mathbf{Z}$ and $H_n(K) = 0$ otherwise (since this is the cone on the (k-1)-simplex).

Definition. Let f, g: K → L be simplicial maps. f and g are contiguous if, for every simplex $\sigma \in K$, there exists a simplex $\tau \in L$ which contains $f(\sigma)$ and $g(\sigma)$.

Proposition. If f and g are contiguous, they are chain homotopic.

Definition. If K is a simplicial complex, we have the augmented chain: … → $C_2(K) \to C_1(K) \to C_0(K) \to \mathrm{Span}\{\varnothing\} = \mathbf{Z} \to 0$, where $\varepsilon: C_0(K) \to \mathrm{Span}\{\varnothing\}$ is defined by $\varepsilon(v) = 1$. (We write these as $C_n\sim(K)$.) The resulting homology groups are the reduced homology groups of K.

Defintion. Let K' be the simplicial complex obtained from K be a single stellar subdivision (this adds a single barycenter vertex, v). We define χ: C(K) → C(K') by $\chi(v_0, \ldots, v_k) = \sum(-1)^i(v, v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$ if v is added to a face of $(v_0, \ldots, v_k)$ and $\chi(S) = S$ otherwise.

Definition. Let θ: K' → K be the simplicial map that sends the added vertex to a vertex in its simplex.

Theorem. θ and χ are chain homotopy inverses.

Lemma. Suppose X is a simplicial complex, A and B subcomplexes with union X, and $N = A \cap B$. Consider $0 \to Cn(N) \to C_n(A) \oplus C_n(B) \to C_n(X) \to 0$, with $\phi_n: C_n(N) \to C_n(A) \oplus C_n(B)$ defined by $\phi_n(c) = (i_A(c), -i_B(c))$ and $\psi_n: C_n(A) \oplus C_n(B) \to C_n(X)$ defined by $\psi_n(c, d) = j_A(c) + j_B(d)$, where i_A, i_B are inclusions from N, and j_A, j_B are inclusions into X. This sequence is exact.

Theorem (Mayer-Vietoris). Let A and B be subcomplexes of X. Let $X = A \cup B$ and $N = A \cap B$. Then the following sequence is exact: $\ldots \to H_nN \to H_nA \oplus H_nB \to H_nX \to H_{n-1}N \to \ldots \to H_0X \to 0$.

Definition. Let K be a simplicial complex. Let α_q be the number of q-simplices in K. We define the Euler characteristic of K by $\chi(K) = \sum (-1)^q \alpha_q$.

Note. If we puncture a surface, we remove a single 2-simplex, which decreases the Euler characteristic by 1. If we puncture two surfaces and then glue them together along the missing triangle, this decreases the number of 2-simplices by 2, decreases the number of 1-simplices by 3 and decreases the number of 3-simplices by 3; this means the Euler characteristic for the connected sum is the sum of the Euler characteristics minus 2.

Theorem. Let K be a simplicial complex. Then, $\chi(K) = \Sigma(-1)^q \beta_q$.

Definition. Let K be a triangulation of S^n, h: |K| → S^n. Let [z] be a generator of $H_n(S^n) = \mathbf{Z}$. Given a continuous map f: $S^n \to S^n$, the induced homomorphism, $(h^{-1}fh)_*: H_nK \to H_nK$ maps [z] to $\lambda[z]$. We call λ the <u>degree</u> of f.

Facts. Homotopic maps have the same degree. The degree of the identity is 1. The degree of a homeomorphism is ± 1. The degree of the constant map is 0. The degree of the antipodal map is $(-1)^{n+1}$. The degree of a map with no fixed points is $(-1)^{n+1}$. $\deg(f \circ g) = \deg(f)\deg(g)$.

Definition. $C_q(K, \mathbf{Z}_2)$ is the set of all formal linear combinations of q-simplices in K with coefficients in $\mathbf{Z}_2$. This induces the <u>mod-2 homology</u>.

Note. The boundary of a simplex in the mod-2 homology is the sum of its q-1 dimensional faces.

Theorem (Borsuk-Ulam). Any map f: $S^n \to \mathbf{R}^n$ must identify a pair of antipodal points of S^n (ie., f(x) = f(-x)).

Theorem. Let f: $S^n \to S^n$ be a map which preserves antipodal points. Then f has odd degree.

Theorem. If f: $S^m \to S^n$ sends antipodal points to antipodal points, $m \leq n$.

Definition. If X is a compact triangulable space, with f: X → X, fix a triangulation h: |K| → X and a simplicial approximation to $f^h = h^{-1}fh$, s: $|K^m| \to |K|$. Let χ: C(K, **Q**) → C(K^m, **Q**) be the subdivision chain map. Then f^h induces a map: $f^h{}_q = s_q \circ \chi_q$: $C_q(K) \to C_q(K)$ for each q, and thus homomorphisms $f^h{}_{q*}$: $H_q(K, \mathbf{Q}) \to H_q(K, \mathbf{Q})$. We may consider this as a linear map of vector spaces. We define the <u>Lefschetz number</u> as $\Lambda_f = \Sigma(-1)^q \text{ trace}(f^h{}_{q*})$. (Note that if $[\sigma_1], \ldots, [\sigma_n]$ is a basis and $f^h{}_{q*}([\sigma_i]) = \ldots + \lambda_i[\sigma_i] + \ldots$, $\text{trace}(f^h{}_{q*}) = \Sigma\, \lambda_i$.)

Theorem (Hopf Trace). If ϕ: C(K, **Q**) → C(K, **Q**) is a chain map, then $\Sigma(-1)^q \text{ trace } \phi_q = \Sigma\, (-1)^q \text{ trace } \phi_{q*}$.

Theorem (Lefschetz Fixed Point). If $\Lambda_f \neq 0$ then f has a fixed point.

Proposition. If g is the constant map, $\Lambda_g = 1$.

Corollary. Homotopic maps have the same Lefschetz number. So any nulhomotopic map has a Lefschetz number of 1, and thus a fixed point.

Theorem. If X is a compact, triangulable space with the homology type of one point space, then every map f: X → X has a fixed point.

Theorem. Let f: $S^n \to S^n$. Then $\Lambda_f = (-1)^n(\deg f) + 1$.

Definition. Let X be a compact Hausdorff space. Take a finite open cover, *F*, of X. Define a simplicial complex called the <u>nerve</u> of *F*, N(*F*), by letting the vertices be the elements of *F* and adding the simplex $(U_0, \ldots, U_n)$ if $U_0 \cap \ldots \cap U_n \neq \emptyset$.

Note. If *F* is the cover of |K| be open stars of K, then N(*F*) is isomorphic to K.

Definition. We say X is <u>finite-dimensional</u> if there exists $m \in \mathbf{Z}$ such that every open cover of X has a refinement, *F*, with dim(N(*F*)) ≤ m. The <u>dimension</u> of X is the smallest such m.

Note. N(*F*) has dimension m ⇔ m+1 is the largest integer such that some m+1 elements of *F* have a non-empty intersection.

Note. The dimension of a simplex (the largest k, such that there is a k-simplex) agrees with this definition.

Theorem. If A is a compact, Hausdorff subspace of X, then dim A ≤ dim X.

Theorem. Let A, B, X be compact Hausdorff, with $X = A \cup B$. Then, dim X = max{dim A, dim B}.

Theorem (Imbedding). Every compact metrizable space of dimension m can be imbedding in $\mathbf{R}^{2m+1}$.

Fact. If F covers X, then $|N(F \cup \{\emptyset\}| = |N(F)| \cup \{p\}$.
Fact. If F covers X, then $N(F \cup \{X\}) = C(N(F))$.

Examples of Topologies

Countable Complement Topology: {U | X – U is countable of all of X}
Discrete Topology: All subsets are open.
Indiscrete Topology: Only X and $\emptyset$ are open.
Standard Topology on **R**: {(a, b) | a < b} is a basis
Standard Topology on $\mathbf{R}^2$: Open balls or open rectangles are a basis.
Lower Limit Topology on **R**: {[a, b)} is a basis
K-Topology on **R**: K = {1/n}. {(a, b)} $\cup$ {(a, b) – K} is a basis
Subspace Topology
Product Topology
Dictionary Order topology on two ordered sets (impose the order (x, y) < (x', y') when X < x' or x = x' and y < y'. Use the order topology on that
The Long Line: $S_\Omega \times [0, 1)$, minus the lowest point, in the dictionary order (S_Ω is an uncountable well-ordered set, every section of which is countable) – every point has a neighborhood homeomorphic to an interval of the real line, but the long line is not homeomorphic to **R**.
The infinite broom: Connected every point (q, 0), q rational, to (1, 0) with a line.
The infinite comb: The interval [0, 1] with spikes up at 0 and each 1/n.
Cantor's Leaky Tent
The Infinite Cage
The Hawaiian earring

Fundamental and First Homology Groups

Theorem. The fundamental group of S^1 is isomorphic to **Z** under +.
Theorem. The fundamental group of $T = S^1 \times S^1$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.
Theorem. Let X be the n-fold connected sum of tori. Then $H_1(X)$ is a free abelian group of rank 2n.
Theorem. Let X be the n-fold connected sum of projective planes. Then $H_1(X)$ is the free abelian product of a group of order 2 and a free abelian group of rank n-1.

Examples of Covering Spaces

S^1, by **R**
P^2, by S^2
The figure 8, by the infinite antenna, or by a line with circles cglued to it

Some Homologies

The point: $H_0(*) = \mathbf{Z}$, $H_n(*) = 0$ otherwise.
S^n (n > 0): $H_n(S^n) = H_0(S^n) = \mathbf{Z}$. $H_k(S^n) = 0$ otherwise.
P^2: $H_1(P^2) = \mathbf{Z}_2$, $H_0(P^2) = \mathbf{Z}$, $H_k(P^2) = 0$ otherwise
Torus: $H_2(T) = H_0(T) = \mathbf{Z}$, $H_1(T) = \mathbf{Z} \oplus \mathbf{Z}$, $H_k(T) = 0$ otherwise.