Analysis Seminar (Shorter) **Summary**

<u>Inverse and Implicit Function Theorems</u>

Lemma. Let A be open in $\mathbf{R}^n$, f: A → $\mathbf{R}^n$ be C^1. If Df(**a**) is non-singular then there exists $\alpha > 0$ and $\varepsilon > 0$ such that $|f(x_1) - f(x_2)| \geq \alpha\, |x_1 - x_2|$ for all x_1, x_2 in C(**a**, ε).

Inverse Function Theorem. Let A be open in $\mathbf{R}^n$ and f: A → $\mathbf{R}^n$ be C^r. If Df is non-singular then there is a neighborhood, U, about **x** and a neighborhood, V, about f(x) such that f: U → V is onto V. Then f^{-1} is C^r.

Implicit Function Theorem. Let A be open in $\mathbf{R}^{n+k}$. Let f: A → $\mathbf{R}^n$ be C^r. Write f as f(**x**, **y**), where $\mathbf{x} \in \mathbf{R}^k$ and $\mathbf{y} \in \mathbf{R}^n$. Let $(\mathbf{a}, \mathbf{b}) \in A$ with f(**a**, **b**) = 0 and det $\partial f/\partial \mathbf{y}\,(\mathbf{a}, \mathbf{b}) \neq 0$. Then, there exists a unique g: $\mathbf{R}^k \to \mathbf{R}^n$ such that f(**a**, g(**a**)) = 0.

<u>Integration Theory</u>

Definition. The <u>volume</u> of Q is given by $v(Q) = (b_1 - a_1) \ldots (b_n - a_n)$.

Definition. The lower and upper sums of a partition, *P*, where each R is a subrectangle of *P* are $L(f; P) = \sum_R m_R v(R)$ and $U(f; P) = \sum_R M_R v(R)$.

Definition. The upper and lower integrals and $\underline{\int}_Q f = \sup\{L(f; P)\}$ and $\overline{\int}_Q f = \inf\{U(f;P)\}$.

Definition. f is <u>integrable</u> over Q if $\underline{\int}_Q f = \overline{\int}_Q f$. We set $\int_Q f$ equal to this value.

Theorem. Let $Q \subset \mathbf{R}^n$ and f: Q → **R** be bounded. Let D be the set of discontinuities of f on Q. Then, f is integrable over Q if and only if D has measure 0.

Definition. Let $S \subset \mathbf{R}^n$ be bounded and f: S → **R** be bounded. Let f_S: $\mathbf{R}^n \to \mathbf{R}$ be defined by f(**x**) = f(**x**) if $\mathbf{x} \in S$ and f(**x**) = 0 elsewhere. We define $\int_S f$ by $\int_Q f_S$ where Q is any rectangle containing S.

Definition. Let S be bounded in $\mathbf{R}^n$. If the constant function 1 is integrable over S, we say S is <u>rectifiable</u>.

Theorem. S is rectifiable if and only if S is bounded and Bd S has measure 0.

Theorem. Let $S \subset \mathbf{R}^n$ be rectifiable. Let f: S → **R** be bounded and discontinuous only on a set of measure 0. Then, $\int_S f$ exists.

Theorem. Let A be open in $\mathbf{R}^n$. Let f: A → **R** be continuous. Choose a sequence, $\{C_n\}$ of compact rectifiable subsets of A whose union is A, such that $C_n \subset \text{Int}\, C_{n+1}$. f is integrable over A if and only if $\{\int_{C_n} |f|\}$ is bounded. Then, $\int_A f = \lim \int_{C_n} f$.

<u>Partitions of Unity</u>

Definition. If ϕ: $\mathbf{R}^n \to \mathbf{R}$ the <u>support</u> of ϕ is the closure of $\{\mathbf{x} \mid \phi(\mathbf{x}) \neq 0\}$.

Theorem. Let *A* be a collection of sets in $\mathbf{R}^n$. Let A be their union. There exists $\{\phi_i\}$, ϕ_i: $\mathbf{R}^n \to \mathbf{R}$ which fulfill the following conditions:

1. $\phi(\mathbf{x}) \geq 0$ for all **x**.
2. S_i = Support $\phi_i \subset A$ for all ϕ_i.
3. If $\mathbf{x} \in A$ then **x** is contained in a finite number of S_i.
4. $\sum \phi_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in A$.
5. Each ϕ_i is C^∞.
6. The S_i are compact.
7. Each S_i is contained in one element of *A*.

Definition. A set of functions fulfilling the first four conditions is called a <u>partition of unity</u>.

Theorem. Let $A \subset \mathbf{R}^n$ be open. Let f: A → **R** be continuous. Let $\{\phi_i\}$ be a partition of unity on A with compact supports. The integral $\int_A f$ exists if and only if $\sum (\int_A \phi_i |f|)$ converges. In this case, $\int_A f = \sum (\int_A \phi_i f)$.

Change of Variables Theorem

Definition. Let A be open in $\mathbf{R}^n$, g: A → $\mathbf{R}^n$ be one-to-one and of class C^r, with det $Dg(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$. Then, g is called a change of variables or a diffeomorphism.

Change of Variables Theorem. Let g: A → B be a diffeomorphism of open sets in $\mathbf{R}^n$. Let f: B → **R** be continuous. Then, f is integrable over B if and only if $(f \circ g)|\det Dg|$ is integrable over A. In that case, $\int_B f = \int_A (f \circ g)|\det Dg|$.

Manifolds

Theorem. There is a unique function, V, that assigns to every k-tuple of elements in $\mathbf{R}^n$ a non-negative number such that (1) If h: $\mathbf{R}^n \to \mathbf{R}^n$ is an isometry then $V(h(\mathbf{x}_1), \ldots, h(\mathbf{x}_n)) = V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, and (2) If $\mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbf{R}^k \times \mathbf{0}^{n-k} \subset \mathbf{R}^n$, so that $\mathbf{y}_i = [\mathbf{z}_i\ \mathbf{0}]$ then $V(\mathbf{y}_1, \ldots, \mathbf{y}_k) = |\det [\mathbf{z}_1 \ldots \mathbf{z}_n]|$. In addition, $V(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ if and only if $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ are dependent. Notice that V is defined by $V = (\det X^T X)^{1/2}$ where $X = [\mathbf{x}_1 \ldots \mathbf{x}_k]$. We call this volume. Also, $V(X) = (\sum_{[I]} (\det X_I)^2)^{1/2}$ where [I] is the set of all ascending k-tuples from {1, 2, …, n}.

Definition. Let k > 0. Let $M \subset \mathbf{R}^n$. Suppose that for any $\mathbf{p} \in M$ there exists V containing **p** such that V is open in M and there exists an open set $U \subset \mathbf{R}^k$ and a one-to-one and onto α: U → V such that (1) α is C^r, (2) α^{-1}: V → U is continuous, and (3) $D\alpha$ has rank k everywhere on U. Then we call α a coordinate patch and M is a k-manifold without boundary.

Theorem. Let M be a k-manifold in $\mathbf{R}^n$ of class C^r. Let $\alpha_0: U_0 \to V_0$, $\alpha_1: U_1 \to V_1$ be coordinate patches on M with $W = V_0 \cap V_1 \neq \emptyset$. Let $W_i = \alpha_i^{-1}(W)$. Then $\alpha_1^{-1} \circ \alpha_0: W_0 \to W_1$ is C^r and $D(\alpha_1^{-1} \circ \alpha_0)$ is non-singular.

Definition. Let M be a k-manifold in $\mathbf{R}^n$. Let $\mathbf{p} \in M$. If there is a coordinate patch, α: U → V on M about **p** that is open in $\mathbf{R}^k$ then **p** is an interior point. If there is no such coordinate patch, then **p** is a boundary point. The set of all boundary points is ∂M.

Scalar Functions on Manifolds.

Definition. Let M be a k-manifold in $\mathbf{R}^n$. Let f: M → **R**. Suppose Support(f) $\subset$ V and α: U → V is a coordinate patch. Them, we define $\int_M f = \int_U (f \circ \alpha)\, V(D\alpha)$.

Definition. Let M be a k-manifold in $\mathbf{R}^n$ and f: M → **R**. Let *V* be a collection of coordinate patches on M. Choose a partition of unity on $\mathbf{R}^n$ dominated by *V* (by extending each V ∈ *V* to an open set in $\mathbf{R}^n$). Since M is compact, all but finitely many ϕ_i vanish at any point of M. Then, $\int_M f = \sum \int_M \phi_i f$.

Tensors, Transformations, and Differential Forms

Definition. f: V^k → **R** is a tensor if f is multi-linear (linear in the i^th coordinate, when all other coordinates are fixed, for all i). The set of all k-tensors on V is $\mathcal{L}^k(V)$.

Theorem. There is a unique ϕ_I: V^k → **R** such that for all $J = (j_1, \ldots, j_k)$, $\phi_I(a_{j1}, \ldots, a_{jk}) = 1$ if I = J, 0 if $I \neq J$. $\{\phi_I\}$ is a basis for $\mathcal{L}^k(V)$ (which is a vector space).

Definition. Let f be a k-tensor and g an l-tensor over the same vector space. The tensor product is defined by $f \otimes g(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})$.

Theorem. If ϕ_I is a k-tensor with $I = (i_1, \ldots, i_k)$, then $\phi_I = \phi_{i1} \otimes \ldots \otimes \phi_{ik}$, where ϕ_{ij} is a 1-tensor.

Definition. A k-tensor on V is alternating if $f(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \mathbf{v}_i, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_n) = - f(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. The set of all alternating k-tensors is $A^k(V)$.

Theorem. Let V be a vector space with basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let $I = (i_1, \ldots, i_k)$ be an ascending k-tuple. There exists a unique k-tensor, Ψ_I, on V, such that for all ascending k-tuples, J, $\Psi_I(\mathbf{a}_{j1}, \ldots, \mathbf{a}_{jk}) = 1$ if I = J and 0 otherwise. These tensors form a basis for $A^k(V)$.

Definition. We define a linear transformation A: $L^k(V) \rightarrow L^k(V)$ by $Af = \sum_\sigma (\operatorname{sgn} \sigma) f^\sigma$. If f is an alternating k-tensor and g is an alternating l-tensor on V, we define the wedge product, an alternating k+l tensor on V, by $f \wedge g = A(f \otimes g)/ k!\, l!$.

Note. $\psi_I = A\phi_I = \phi_{i1} \wedge \ldots \wedge \phi_{ik}$.. $\Psi_I(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \det X_I$.

Theorem. The wedge product is associative, homogeneous, and distributive. In addition, $g \wedge f = (-1)^{kl} f \wedge g$. The wedge product is preserved under the pullback: $T^*(f \wedge g) = (T^*f) \wedge (T^*g)$.

Definition. Let $\mathbf{x} \in \mathbf{R}^n$. A tangent vector to $\mathbf{R}^n$ at $\mathbf{x}$ is $(\mathbf{x}; \mathbf{v})$, where $\mathbf{v} \in \mathbf{R}^n$. We define: $(\mathbf{x}; \mathbf{v}) + (\mathbf{x}; \mathbf{w}) = (\mathbf{x}; \mathbf{v} + \mathbf{w})$ and $c(\mathbf{x}; \mathbf{v}) = (\mathbf{x}; c\mathbf{v})$. The set of all tangent vectors to $\mathbf{R}^n$ at $\mathbf{x}$ is called the tangent space to $\mathbf{R}^n$ at $\mathbf{x}$, or $T_\mathbf{x}(\mathbf{R}^n)$.

Definition. Let A be open in $\mathbf{R}^k$ or $\mathbf{H}^k$, α: A $\rightarrow \mathbf{R}^n$ be of class C^r. Let $\mathbf{x} \in$ A and $\mathbf{p} = \alpha(\mathbf{x})$. We define $\alpha_*: T_\mathbf{x}(\mathbf{R}^k) \rightarrow T_\mathbf{p}(\mathbf{R}^n)$ by $\alpha_*(\mathbf{x}; \mathbf{v}) = (\mathbf{p}; D\alpha(\mathbf{x}) * \mathbf{v})$. This is the transformation induced by α and a push-forward.

Lemma. Let A be open in $\mathbf{R}^k$ or $\mathbf{H}^k$. Let α: A $\rightarrow \mathbf{R}^m$ be C^r. Let B be open in $\mathbf{R}^m$ or $\mathbf{H}^m$, with $\alpha(A) \subset B$. Let β: B $\rightarrow \mathbf{R}^n$ be C^r. Then, $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.

Definition. Let M be a k-manifold of class C^r in $\mathbf{R}^n$. If $\mathbf{p} \in$ M, choose a coordinate patch α: U $\rightarrow$ V about $\mathbf{p}$. Let $\mathbf{x} \in$ U such that $\alpha(\mathbf{x}) = \mathbf{p}$. Then we define the tangent space to M at $\mathbf{p}$ by $T_\mathbf{p}(M) = \alpha_*(T_\mathbf{x}(\mathbf{R}^k)) = \{\alpha_*(\mathbf{x}; \mathbf{v}) \mid \mathbf{v} \in \mathbf{R}^k\}$.

Definition. Let $A \subset \mathbf{R}^n$ be open. A tangent vector field in A is a continuous function F: A $\rightarrow \mathbf{R}^n \times \mathbf{R}^n$ such that $F(\mathbf{x}) \in T_\mathbf{x}(\mathbf{R}^n)$. Thus, we may write $F(\mathbf{x}) = (\mathbf{x}; f(\mathbf{x}))$ where f: A $\rightarrow \mathbf{R}^n$.

Definition. Let $A \subset \mathbf{R}^n$ be open. A k-tensor field. in A is ω: $\mathbf{x} \rightarrow L^k(T_\mathbf{x}(\mathbf{R}^n))$; in other words, ω assigns a k-tensor defined on $T_\mathbf{x}(\mathbf{R}^n)$ to each $\mathbf{x} \in$ A. Notice that $\omega(\mathbf{x})((\mathbf{x}; \mathbf{v}_1), \ldots, (\mathbf{x}; \mathbf{v}_k))$ must be continuous as a function of $\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_k$. If $\omega(\mathbf{x})$ is an alternating k-tensor for all $\mathbf{x}$, we call ω a differential form of order k on A.

Definition. The elementary 1-forms on $\mathbf{R}^n$ are given by $\phi_i(\mathbf{x})(\mathbf{x}; \mathbf{e}_j) = 1$ if i = j, 0 otherwise. The elementary k-forms on $\mathbf{R}^n$ are given by $\psi_I(\mathbf{x}) = \phi_{i1}(\mathbf{x}) \wedge \ldots \wedge \phi_{ik}(\mathbf{x})$, where $I = (i_1, \ldots, i_k)$ is an ascending k-tuple.

Note. If ω is a k-form on A, we may write $\omega(\mathbf{x}) = \sum_{[I]} b_I(\mathbf{x})\, \psi_I(\mathbf{x})$, where the b_I are scalar functions and are called the components of ω.

Definition. Let A be open in $\mathbf{R}^n$. If f: A $\rightarrow \mathbf{R}$ is C^r, f is called a scalar field in A and a differential form of order 0.

Note. $\omega(\mathbf{x}) \wedge f(\mathbf{x}) = f(\mathbf{x})\, \omega(\mathbf{x})$.

Definition. Let A be open in $\mathbf{R}^n$ and f: A $\rightarrow \mathbf{R}$ be C^∞. The, $d(f(\mathbf{x}; \mathbf{v})) = Df(\mathbf{x}) \bullet \mathbf{v}$. We call this the differential of f.

Theorem. Let A be open in $\mathbf{R}^n$ and f: A → **R** be C^∞. Then, $df = (D_1f)dx_1 + \ldots + (D_nf)dx_n$.

Definition. Let $\omega \in \Omega^k(A)$, j > 0. Let $\omega = \sum_{[I]} f_I\, dx_I$. We define $d\omega = \sum_{[I]} df_I \wedge dx_I$.

Theorem. Let d: $\Omega^k(A) \to \Omega^{k+1}(A)$.

- d is linear ($d(a\omega + b\eta) = a(d\omega) + b(d\eta)$.).
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\, \omega \wedge d\eta$, where ω is a k-form and η is any form.
- $d(d\omega) = 0$ for all forms.

Definition. A form, ω, is <u>closed</u> if dω = 0.

Definition. A form, ω, is <u>exact</u> if ω = dθ for some k-1 form θ.

Definition. Let B be open in $\mathbf{R}^n$ and $\alpha(A) \subset B$. A <u>dual transformation of forms</u> (<u>pullback</u>) is given by $(\alpha^*f)(\mathbf{x}) = f(\alpha(\mathbf{x}))$ if f is a 0 form, and $((\alpha^*\omega)(\mathbf{x}))(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \omega(\alpha(\mathbf{x}))(\alpha_*(\mathbf{x}; \mathbf{v}_1), \ldots, \alpha_*(\mathbf{x}; \mathbf{v}_n))$.

Proposition. Let ω, η, and θ be forms, with ω and η having the same order. Then:

- $\alpha^*(a\omega + b\eta) = a\,\alpha^*(\omega) + b\,\alpha^*(\eta)$ [linear]
- $\alpha^*(\omega \wedge \theta) = \alpha^*(\omega) \wedge \alpha^*(\theta)$
- $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$
- $\alpha^*(d\omega) = d(\alpha^*\omega)$.

Theorem. Let A be open in $\mathbf{R}^k$. Let α: A → $\mathbf{R}^n$ be C^∞. Let $\mathbf{x} \in \mathbf{R}^k$ and $\mathbf{y} \in \mathbf{R}^n$ with $\alpha(\mathbf{x}) = \mathbf{y}$. If $I = (i_1, \ldots, i_l)$ is an ascending l-tuple from {1, 2, …, n} then $\alpha^*(dy_I) = \sum_{[J]} \det(\partial\alpha_I / \partial x_J)\, dx_J$.

<u>Integrating over a Manifold.</u>

Note. $\int_A f\, dx_1 \wedge .. dx_n = \int_A f$, for $A \subset \mathbf{R}^n$.

Theorem. Let $\omega = f\, dz_I$. Then, $\int_{Y\alpha} \omega = \int_A \alpha^*\omega = \int_A (f \circ \alpha) \det(\partial\alpha_I/\partial x)$.

Definition. Let M be a k-manifold in $\mathbf{R}^n$. Let $\alpha_0: U_0 \to V_0$ and $\alpha_1: U_1 \to V_1$ be coordinate patches on M. If $V_0 \cap V_1$ is non-empty, then α_0 and α_1 <u>overlap</u>. If α_0 and α_1 overlap and $\alpha_1^{-1} \circ \alpha_0$ is orientation-preserving, then α_0 and α_1 <u>overlap positively</u>. If α_0 and α_1 overlap and $\alpha_1^{-1} \circ \alpha_0$ is not orientation-preserving, then α_0 and α_1 <u>overlap negatively</u>. If we may cover M by coordinate patches that overlap positively or not at all, we call M <u>orientable</u>.

Definition. Let M be a compact oriented k-manifold in $\mathbf{R}^n$. Let ω be a k-form defined in an open set of $\mathbf{R}^n$ containing M. Let C = M ∩ Support ω; note that C is compact. Suppose there is a coordinate patch α: U → V on M belonging to the orientation of M with C ⊂ V. Assume U is bounded. We define the integral of ω over M as $\int_M \omega = \int_{\text{Int } U} \alpha^*(\omega)$.

Definition. Let M be a compact oriented k-manifold in $\mathbf{R}^n$. Let ω be a k-form defined in an open set of $\mathbf{R}^n$ containing M. Cover M by coordinate patches belonging to the orientation of M; choose a partition of unity on M dominated by these coordinate patches. We define $\int_M \omega = \sum (\int_M \phi_i\, \omega)$.

Definition. A <u>0-manifold</u> is a finite collection of points, $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ in $\mathbf{R}^n$. We define an orientation on such a manifold by a function $\varepsilon: \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \to \{-1, 1\}$. $\int_M f = \sum \varepsilon(\mathbf{x}_i)f(\mathbf{x}_i)$.

Definition. Let M be an oriented 1-manifold in $\mathbf{R}^n$. We define the orientation of ∂M by $\varepsilon(\mathbf{p}) = -1$ if there exists α: U → V, about **p**, with $U \subset \mathbf{H}^k$ and $\varepsilon(\mathbf{p}) = 1$ otherwise.

Stokes Theorem. Let M be an oriented k manifold in $\mathbf{R}^n$. Let ∂M have the induced orientation. Let ω be a k-1 form on an open set containing M. Then, $\int_M d\omega = \int_{\partial M} \omega$.

Lebesgue Measure

Definition. A σ-algebra, or a Borel field, is an algebra of sets that is closed under countable union (and therefore countable intersection). A Borel set is the smallest σ-algebra that contains the closed and open intervals.

Definition. The outer measure of a set $E \subset \mathbf{R}$ is $m^*E = \inf_{E \subset \cup I(n)} \sum l(I_n)$ (where l(I) is the length of the interval), so that $\{I_n\}$ is a set of intervals that covers E.

Proposition. Let $\{A_n\}$ be any countable collection of sets. Then, $m^*(\cup A_n) \leq \sum m^*A_n$. (This is called countable subadditivity.)

Proposition. Given any set A and $\varepsilon > 0$, there exists an open set O such that $A \subset O$ and $m^*O \leq m^*A + \varepsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.

Definition. E is measurable if, for all A, $m^*A = m^*(A \cap E) + m^*(A \cap E^C)$.

Lemma. If $m^*E = 0$, E is measurable.

Theorem. The measurable sets form an algebra.

Lemma. Let A be any set. Let $E_1, \ldots, E_n$ be a finite sequence of disjoint measurable sets. Then, $m^*(A \cap (\cup E_i)) = \sum m^*(A \cap E_i)$.

Lemma. (a, ∞) is measurable.

Theorem. Every Borel set is measurable.

Proposition. Let $\{E_i\}$ be an infinite, decreasing (ie. $E_{i+1} \subset E_i$) sequence of measurable sets. Let $mE_1 < \infty$. Then, $m(\cap E_i) = \lim mE_n$.

Proposition. Let E be a set. The following are equivalent:

i. E is measurable.

ii. Given $\varepsilon > 0$ there exists an open set $O \supset E$ with $m^*(O - E) < \varepsilon$.

iii. Given $\varepsilon > 0$ there exists a closed set $F \subset E$ with $m^*(E - F) < \varepsilon$.

iv. There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G - E) = 0$.

v. There is an $F \in F_\sigma$ with $F \subset E$ such that $m^*(E - F) = 0$.

vi. (If $m^*E < \infty$, then) If $\varepsilon > 0$ there exists a *finite* union $U = \cup I_i$ such that $m^*((U - E) \cup (E - U)) < \varepsilon$.

Proposition. Let f be any function. Let $a \in \mathbf{R}$. The following are equivalent:

i. $\{x \mid f(x) > a\}$ is measurable.

ii. $\{x \mid f(x) < a\}$ is measurable.

iii. $\{x \mid f(x) \leq a\}$ is measurable.

iv. $\{x \mid f(x) \geq a\}$ is measurable.

Definition. A function is measurable if its domain is measurable and the conditions above hold.

Theorem. Let $\{f_n\}$ be a sequence of measurable functions defined on the same domain. Then the functions $\sup\{f_1, \ldots, f_n\}$, $\inf\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\limsup f_n$, and $\liminf f_n$ are also measurable.

Theorem. If f is measurable and f = g almost everywhere, then g is measurable.

Littlewood's Three Principles. Every measurable set is nearly a union of intervals. Every measurable function is nearly continuous. Every convergent sequence of measurable functions is nearly uniformly convergent.

Lebesgue Integration

Definition. A simple function, φ, is defined by $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$, where χ_{E_i} is the characteristic function of E_i (1 on E_i, 0 elsewhere).

Note. φ is simple if and only if it is measurable and takes on a finite number of values.

Definition. Let $\varphi = \sum a_i \chi_{E_i}$. We define $\int \varphi = \sum a_i mE_i$.

Definition. If f is bounded and measurable on a set E of finite measure, we define $\int_E f = \inf_{\psi \geq f} \int_E \psi = \sup_{\varphi \leq f} \int_E \varphi$.

Proposition. If f and g are bounded, measurable functions defined on a set E of finite measure, then:

- $\int_E (af + bg) = a\int_E f + b\int_E g$.
- If $g \leq f$ almost everywhere, then $\int_E g \leq \int_E f$.
- If $A \leq f(x) \leq B$ almost everywhere, then $A(mE) \leq \int_E f \leq B(mE)$.
- If $A \cap B = \varnothing$ and A, B have finite measure, $\int_{A \cup B} f = \int_A f + \int_B f$.

Definition. If f is non-negative and measurable on any measurable set E, we define $\int_E f = \sup_{h \leq f} \int_E h$ where h is bounded, measurable, and non-zero only on a set of finite measure. f is integrable over the measurable set E if $\int_E f < \infty$.

Definition. Let f be any function. Define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\} = \max\{-f(x), 0\}$.

Definition. Let f be measurable. f is integrable over E if f^+ and f^- are integrable. Then, $\int_E f = \int_E f^+ - \int_E f^-$.

Proposition. The following properties hold for the general Lebesgue integral:

- $\int_E (af + bg) = a\int_E f + b\int_E g$.
- If $g \leq f$ almost everywhere, then $\int_E g \leq \int_E f$.
- If $A \cap B = \varnothing$ and A, B have finite measure, $\int_{A \cup B} f = \int_A f + \int_B f$.

Bounded Convergence Theorem. Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure. Suppose there is some real number M such that $|f_n(x)| \leq M$ for all n, x. If $f(x) = \lim f_n(x)$ almost everywhere in E, then $\int_E f = \lim \int_E f_n$.

Fatou's Lemma. Let $\langle f_n \rangle$ be a sequence of non-negative, measurable functions with $\lim f_n(x) = f(x)$ almost everywhere on a measurable set E. Then, $\int_E f \leq \underline{\lim} \int_E f_n$.

Monotone Convergence Theorem. Let $\langle f_n \rangle$ be an increasing sequence of non-negative, measurable functions with $f(x) = \lim f_n(x)$ almost everywhere. Then, $\int f = \lim \int f_n$.

Lebesgue Convergence Theorem. Let g be integrable over E and $\langle f_n \rangle$ a sequence of measurable functions with $|f_n| \leq g$ everywhere on E. Let $\lim f_n(x) = f(x)$ almost everywhere on E. Then, $\int_E f = \lim \int_E f_n$.

Theorem. Let $\langle g_n \rangle$ be a sequence of measurable functions over E that converge almost everywhere to an integrable function g. Let $\langle f_n \rangle$ be a sequence of measurable functions with $|f_n| \leq g_n$ and $\lim f_n(x) = f(x)$ almost everywhere. If $\int_E g = \lim \int_E g_n$ then $\int_E f = \lim \int_E f_n$.

Definition. A sequence $\langle f_n \rangle$ converges to f in measure if, given $\varepsilon > 0$, there exists N such that, for all $n > N$, $m\{x \mid |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon$.

Proposition. Suppose $\langle f_n \rangle$ converges to f in measure and all the f_n are measurable. Then there is a subsequence $\langle f_{nk} \rangle$ that converges to f almost everywhere.

Corollary. Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure. Then f_n converges to f in measure if and only if every subsequence of $\langle f_n \rangle$ has in turn a subsequence that converges to f almost everywhere.

Proposition. The convergence theorems stated above hold if "convergence almost everywhere" is replaced by "convergence in measure."

L^p Spaces

Definition. $f \in L^p$ if $\int_{[0,1]} |f|^p < \infty$.

Definition. $||f||_p = (\int_{[0,1]} |f|^p)^{1/p}$.

Note. If we consider functions equivalent when they are equal almost everywhere, then L^p is a normed linear space.

Definition. $f \in L^\infty$ if f is bounded almost everywhere and measurable. $||f||_\infty$ = ess sup $|f(x)| = \inf \{M \mid m\{t \mid f(t) > M\} = 0\}$.

Applications to Vector Calculus

Definition. Let A be open in $\mathbf{R}^n$. Let f: A $\rightarrow$ $\mathbf{R}$ be a scalar field. We define the gradient of f by (grad f)($\mathbf{x}$) = ($\mathbf{x}$; $D_1 f(\mathbf{x})\mathbf{e}_1 + \ldots + D_n f(\mathbf{x})\mathbf{e}_n$). Let G($\mathbf{x}$) = ($\mathbf{x}$; g($\mathbf{x}$)) be a vector field on A, with $g(\mathbf{x}) = g_1(\mathbf{x})\mathbf{e}_1 + \ldots + g_n(\mathbf{x})\mathbf{e}_n$. We define the divergence of G by (div G) = $D_1 g_1(\mathbf{x}) + \ldots + D_n g_n(\mathbf{x})$.

Theorem. Let A be open in $\mathbf{R}^n$. Then we have the following vector space isomorphisms:

α_0: Scalar fields in A $\rightarrow$ $\Omega^0(A)$.

α_1: Vector fields in A $\rightarrow$ $\Omega^1(A)$

β_{n-1}: Vector fields in A $\rightarrow$ $\Omega^{n-1}(A)$

β_n: Scalar field in A $\rightarrow$ $\Omega^n(A)$

so that $d \circ \alpha_0 = \alpha_1 \circ$ grad and $d \circ \beta_{n-1} = \beta_n \circ$ div.

Proof. $\alpha_0(f) = f$

$\alpha_1(F) = \sum f_i \, dx_i$

$\beta_{n-1}(G) = \sum (-1)^{i-1} g_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots dx_n$

$\beta_n(h) = h \, dx_1 \wedge \ldots \wedge dx_n$

Definition. Let $A \subset \mathbf{R}^3$ be open. Let F($\mathbf{x}$) = ($\mathbf{x}$; $\sum f_i(\mathbf{x})\mathbf{e}_i$) be a vector field in A. We define the vector field, curl F, by (curl F)($\mathbf{x}$) = ($\mathbf{x}$; $(D_2 f_3 - D_3 f_2)(\mathbf{x})\mathbf{e}_1 + (D_3 f_1 - D_1 f_3)(\mathbf{x})\mathbf{e}_2 + (D_1 f_2 - D_2 f_1)(\mathbf{x})\mathbf{e}_3$).

Theorem. Let A be open in $\mathbf{R}^3$. Then, in addition to the isomorphisms in the previous theorem, we find that $d \circ \alpha_1 = \beta_2 \circ$ curl.

Definition. Let M be a 1-manifold. The unit tangent vector at $\mathbf{p} \in$ M if given by $T(\mathbf{p})$ = ($\mathbf{p}$; $D\alpha(t)/||D\alpha(t_0)||$) where $\alpha(t_0) = \mathbf{p}$.

Definition. So that we may have outward-pointing unit tangent vectors, we define the left-half-line, $\mathbf{L} = \{x \mid x \leq 0\}$ and allow coordinate patches α: $\mathbf{L} \rightarrow \mathbf{R}^n$.

Definition. Let M be an n-1 manifold in $\mathbf{R}^n$. Let $\mathbf{p} \in$ M. Let ($\mathbf{p}$; $\mathbf{n}$) be a tangent vector to $\mathbf{R}^n$ that is orthogonal to the tangent space to M at $\mathbf{p}$. Let $||\mathbf{n}|| = 1$. If $\mathbf{n}$ is always pointing the "same" direction, this is called a normal vector field to M and defines an orientation.

Definition. Let M be an n-manifold in $\mathbf{R}^n$. The natural orientation of M is the set of all coordinate patches α: $\mathbf{R}^n \rightarrow \mathbf{R}^n$ with det $D\alpha > 0$.

Theorem. Let k > 1. If M is an oriented k-manifold in $\mathbf{R}^n$ with ∂M non-empty, then ∂M is orientable.

Definition. Let M be an orientable k-manifold in $\mathbf{R}^n$, with ∂M non-empty. Given an orientation of M, the induced orientation of ∂M is defined by the orientation of the restricted coordinate patch if k is even and the opposite orientation if k is odd.

Note. The induced orientation of an n-1 manifold that is the boundary of a naturally oriented n-manifold always points outward from the manifold.

Theorem. Let M be a compact, oriented n-1 manifold in $\mathbf{R}^n$. Let N be the unit normal field (corresponding to the induced orientation). Let G be a vector field on an open set containing M, so that $G(\mathbf{y}) = (\mathbf{y}; g(\mathbf{y})) = (\mathbf{y}; \sum g_i(\mathbf{y})\, \mathbf{e}_i)$. Let $\omega = \sum (-1)^{i-1} g_i\, dy_1 \wedge \ldots \wedge dy_{i-1} \wedge dy_{i+1} \wedge \ldots \wedge dy_n$. Then, $\int_M \omega = \int_M <G, N>\, dV$.

Theorem. Let M be an n-manifold in $\mathbf{R}^n$. Let $\omega = h\, dx_1 \wedge \ldots \wedge dx_n$. Then, $\int_M \omega = \int_M h\, dV$.

Divergence Theorem. Let M be a compact, oriented n-manifold in $\mathbf{R}^n$. Let **N** be the unit normal field. If G is a vector field, then $\int_M (\text{div } G)\, dV = \int_{\partial M} <G, \mathbf{N}>\, dV$.

Classical Stokes Theorem. Let M be a compact, oriented 2-manifold in $\mathbf{R}^3$. Let **N** be the unit normal field. Let **F** be a C^∞ function. Then, if $\partial M = \emptyset$, then $\int_M <\text{curl } F, N>\, dV = 0$. Otherwise, $\int_M <\text{curl } F, N>\, dV = \int_{\partial M} <F, T>\, dV$, where T is the unit tangent field to ∂M with the induced orientation.