Analysis Seminar Summary

Inverse and Implicit Function Theorems

Definition. $C(\mathbf{a}, r)$ is the open cube in $\mathbf{R}^n$ about $\mathbf{a}$ of side length 2r. It is the set of points where each coordinate differs from any coordinate of $\mathbf{a}$ by at most r.

Lemma. Let A be open in $\mathbf{R}^n$, $f: A \to \mathbf{R}^n$ be C^1. If $Df(\mathbf{a})$ is non-singular then there exists $\alpha > 0$ and $\varepsilon > 0$ such that $|f(x_1) - f(x_2)| \geq \alpha |x_1 - x_2|$ for all x_1, x_2 in $C(\mathbf{a}, \varepsilon)$.

Note. This lemma implies that f is 1-1 in a neighborhood of $\mathbf{a}$.

Lemma. Let A be open in $\mathbf{R}^n$ and $f: A \to \mathbf{R}^n$ be C^r. Let B = f(A). If f is 1-1 and $Df(\mathbf{x})$ is non-singular for all $\mathbf{x} \in A$ then B is open in $\mathbf{R}^n$ and f^{-1} is C^r.

Proof. Step 1: If a function has a local minimum or maximum, then the derivative is 0 there.

Step 2: B = f(A) is open.

Step 3: f^{-1} is continuous.

Step 4: f^{-1} is differentiable.

Step 5: f^{-1} is C^r.

Inverse Function Theorem. Let A be open in $\mathbf{R}^n$ and $f: A \to \mathbf{R}^n$ be C^r. If $Df(\mathbf{x})$ is non-singular then there is a neighborhood, U, about $\mathbf{x}$ and a neighborhood, V, about f(x) such that $f: U \to V$ is onto V. Then f^{-1} is C^r.

Lemma. Let A be open in $\mathbf{R}^{n+k}$. Let $f: A \to \mathbf{R}^n$ be differentiable. Suppose there exists $g: \mathbf{R}^k \to \mathbf{R}^n$ such that $f(\mathbf{x}, g(\mathbf{x})) = 0$ for all $\mathbf{x} \in \mathbf{R}^k$. Then, $\partial f/\partial \mathbf{x} + \partial f/\partial \mathbf{y} * \partial g/\partial \mathbf{x} = 0$.

Implicit Function Theorem. Let A be open in $\mathbf{R}^{n+k}$. Let $f: A \to \mathbf{R}^n$ be C^r. Write f as $f(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in \mathbf{R}^k$ and $\mathbf{y} \in \mathbf{R}^n$. Let $(\mathbf{a}, \mathbf{b}) \in A$ with $f(\mathbf{a}, \mathbf{b}) = 0$ and $\det \partial f/\partial \mathbf{y}\ (\mathbf{a}, \mathbf{b}) \neq 0$. Then, there exists a unique $g: \mathbf{R}^k \to \mathbf{R}^n$ such that $f(\mathbf{a}, g(\mathbf{a})) = 0$.

Proof. Apply the inverse function theorem to $F(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, f(\mathbf{x}, \mathbf{y}))$.

Integration Theory

Definition. A rectangle, Q, in $\mathbf{R}^n$ is $Q = [a_1, b_1] \times \ldots \times [a_n, b_n]$.

Definition. The volume of Q is given by $v(Q) = (b_1 - a_1) \ldots (b_n - a_n)$.

Definition. A partition, *P*, of Q is an n-tuple of partitions of $[a_i, b_i]$. The parts of the partition are called subrectangles.

Definition. If $f: Q \to \mathbf{R}$ is bounded, then we define $m_R(f) = \inf\{f(x) \mid x \in R\}$, where R is a subrectangle of Q.

Definition. Under the same conditions, $M_R(f) = \sup\{f(x) \mid x \in R\}$.

Definition. The lower and upper sums of a partition, *P*, where each R is a subrectangle of *P* are $L(f; P) = \sum_R m_R v(R)$ and $U(f; P) = \sum_R M_R v(R)$.

Definition. The upper and lower integrals and $\underline{\int}_Q f = \sup\{L(f; P)\}$ and $\overline{\int}_Q f = \inf\{U(f;P)\}$.

Definition. f is integrable over Q if $\underline{\int}_Q f = \overline{\int}_Q f$. We set $\int_Q f$ equal to this value.

Theorem. Suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that if *P* is any partition of mesh less than δ with $x_R \in R$ then $|\Sigma_R f(x_R)v(R) - A| < \varepsilon$. Then, f is integrable over Q with $\int_Q f = A$.

Definition. Let $A \subset \mathbf{R}^n$. A has measure zero if, for all $\varepsilon > 0$, there exists a covering $\{Q_i\}$ of A by countable many rectangles such that $\sum v(Q_i) < \varepsilon$.

Theorem. Let $Q \subset \mathbf{R}^n$ and $f: Q \to \mathbf{R}$ be bounded. Let D be the set of discontinuities of f on Q. Then, f is integrable over Q if and only if D has measure 0.

Definition. Let $S \subset \mathbf{R}^n$ be bounded and $f: S \to \mathbf{R}$ be bounded. Let $f_S: \mathbf{R}^n \to \mathbf{R}$ be defined by $f(\mathbf{x}) = f(\mathbf{x})$ if $\mathbf{x} \in S$ and $f(\mathbf{x}) = 0$ elsewhere. We define $\int_S f$ by $\int_Q f_S$ where Q is any rectangle containing S.
Lemma. $\int_S f$ is well-defined.
Theorem. Under the conditions of boundedness, the following conditions about the integral hold:

- Linearity: $\int_S (af + bg) = a\int_S f + b\int_s g$ provided the right-hand integrals exist.
- Comparison: If $f(\mathbf{x}) \le g(\mathbf{x})$ for all $\mathbf{x} \in S$ then $\int_S f \le \int_S g$. In particular, $|f|$ is integrable if f is integrable, and $|\int_S f| \le \int_S |f|$.
- Monotonicity: Suppose $T \subset S$, $f(\mathbf{x})$ is non-negative on S, and both $\int_S f$ and $\int_T f$ exist. Then, $\int_T f \le \int_S f$.
- Additivity: Let $S = S_1 \cup S_2$. If f is integrable over S_1 and S_2 then it is integrable over their union and intersection, with $\int_S f = \int_{S1} f + \int_{S2} f - \int_{S1 \cap S2} f$.

Definition. Let S be bounded in $\mathbf{R}^n$. If the constant function 1 is integrable over S, we say S is rectifiable.
Definition. We define the (Jordan) volume of S by $v(S) = \int_S 1$.
Theorem. S is rectifiable if and only if S is bounded and Bd S has measure 0.
Proof (sketch). The function that is 1 on S and 0 elsewhere can be integrated unless the set of discontinuities, which is Bd S, has measure greater than 0.
Theorem. Let $S \subset \mathbf{R}^n$ be rectifiable. Let $f: S \to \mathbf{R}$ be bounded and discontinuous only on a set of measure 0. Then, $\int_S f$.
Lemma. Let A be open in $\mathbf{R}^n$. Let $f: A \to \mathbf{R}$ be continuous. If $C \subset A$ is compact and f vanishes outside C the $\int_A f = \int_C f$.
Theorem. Let A be open in $\mathbf{R}^n$. Let $f: A \to \mathbf{R}$ be continuous. Choose a sequence, $\{C_n\}$ of compact rectifiable subsets of A whose union is A, such that $C_n \subset \text{Int } C_{n+1}$. f is integrable over A if and only if $\{\int_{Cn} |f| \}$ is bounded. Then, $\int_A f = \lim \int_{Cn} f$.

Partitions of Unity

Lemma. Let $Q \subset \mathbf{R}^n$ be a rectangle. Then, there exists a C^∞ function, $\phi: \mathbf{R}^n \to \mathbf{R}$, such that $\phi(x) > 0$ for $\mathbf{x} \in \text{Int } Q$ and $\phi(\mathbf{x}) = 0$ elsewhere.
Proof. Let $f(\mathbf{x}) = e^{-1/x}e^{-1/(1-x)}$. This function works for [0, 1].
Lemma. Let $\mathcal{A}$ be a collection of open sets in $\mathbf{R}^n$. Let $A = \cup \mathcal{A}$. Then, there exists a countable collection, $\{Q_i\}$ of rectangles contained in A such that

1. $\cup Q_i$ covers A
2. Each Q_i is contained in some element of $\mathcal{A}$.
3. Each $a \in A$ has a neighborhood that intersects finitely many Q_i.

Proof (outline). Cover A by compact nested subsets, D_i. Let $B_i = D_i - \text{Int } D_{i-1}$. Cover each B_i by closed cubes that do not intersect D_{i-2}, and thus by a finite subcover. These are the Q_i.
Definition. If $\phi: \mathbf{R}^n \to \mathbf{R}$ the support of ϕ is the closure of $\{\mathbf{x} \mid \phi(\mathbf{x}) \neq 0\}$.
Theorem. Let $\mathcal{A}$ be a collection of sets in $\mathbf{R}^n$. Let A be their union. There exists $\{\phi_i\}$, $\phi_i: \mathbf{R}^n \to \mathbf{R}$ which fulfill the following conditions:

1. $\phi(\mathbf{x}) \ge 0$ for all $\mathbf{x}$.
2. $S_i = \text{Support } \phi_i \subset A$ for all ϕ_i.
3. If $\mathbf{x} \in A$ then $\mathbf{x}$ is contained in a finite number of S_i.

4. $\sum\phi_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in A$.
5. Each ϕ_i is C^∞.
6. The S_i are compact.
7. Each S_i is contained in one element of *A*.

Proof. Use the previously constructed $\{Q_i\}$ and ϕ for each Q_i, normalizing so that they sum to one.

Definition. A set of functions fulfilling the first four conditions is called a partition of unity.

Theorem. Let $A \subset \mathbf{R}^n$ be open. Let $f: A \to \mathbf{R}$ be continuous. Let $\{\phi_i\}$ be a partition of unity on A with compact supports. The integral $\int_A f$ exists if and only if $\sum (\int_A \phi_i |f|)$ converges. In this case, $\int_A f = \sum (\int_A \phi_i f)$.

Change of Variables Theorem

Definition. Let A be open in $\mathbf{R}^n$, $g: A \to \mathbf{R}^n$ be one-to-one and of class C^r, with $\det Dg(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$. Then, g is called a change of variables or a diffeomorphism.

Lemma. Let A be open in $\mathbf{R}^n$ and $g: A \to \mathbf{R}^n$ be C^r. If $E \subset A$ has measure 0 in $\mathbf{R}^n$ then g(E) has measure 0 in $\mathbf{R}^n$ as well.

Proof. Cover E by cubes. Show that g sends each cube to the interior of a cube with width nMw, where w was the width of the old cube and M is the largest absolute value in the matrix det Dg. Cover A with nested compact sets and let E_k be the intersection of E with each. Cover E_k by finitely may cubes with total volume less than $\varepsilon/(nM)^n$. Show that g(E) has measure less than ε.

Theorem. Let $g: A \to B$ be a diffeomorphism of class C^r of open sets in $\mathbf{R}^n$. Let D, Bd $D \subset A$ and E = g(D). Then, g(Int D) = Int E and g(Bd D) = Bd E. If D is rectifiable, sp is E.

Definition. Let $h: A \to B$ be a diffeomorphism of open sets in $\mathbf{R}^n$, $n \geq 2$, with $h(\mathbf{x}) = (h_1(\mathbf{x}), \ldots h_n(\mathbf{x}))$. h preserves the i^{th} coordinate if $h_i(\mathbf{x}) = x_i$ for all $\mathbf{x} \in A$. If h preserves the i^{th} coordinate for any i, then h is a primitive diffeomorphism.

Theorem. Let $g: A \to B$ be a diffeomorphism of open sets in $\mathbf{R}^n$, $n \geq 2$. Then, there is a neighborhood U_0 of **a** and a sequence of primitive diffeomorphisms, $h_1: U_1 \to U_0$, $h_2: U_2 \to U_1, \ldots, h_k: U_k \to U_{k-1}$ such that $h_k \circ \ldots \circ h_1 \circ h_0(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in U_0$.

Theorem (Substitution Rule). Let I = [a, b], $g: I \to \mathbf{R}$ be C_1 with $g'(\mathbf{x}) \neq 0$ for all $x \in (a, b)$. Then, g(I) = J is a closed interval with endpoints g(a) and g(b), If $f: J \to \mathbf{R}$ is continuous then $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g)\, g'$ and $\int_J f = \int_I (f \circ g)|g'|$.

Change of Variables Theorem. Let $g: A \to B$ be a diffeomorphism of open sets in $\mathbf{R}^n$. Let $f: B \to \mathbf{R}$ be continuous. Then, f is integrable over B if and only if $(f \circ g)|\det Dg|$ is integrable over A. In that case, $\int_B f = \int_A (f \circ g)|\det Dg|$.

Proof. Step 1: Prove the case where n = 1, only in a neighborhood of a point. (Use the substitution rule on an interval with **x** in the interior and the support of f contained in the interval.)

Step 2: The case for primitive diffeomorphisms with n > 1, locally. (Using primitive-ness, reduce the problem to a question of n – 1.)

Step 3: Composing primitive diffeomorphism, using the chain rule.

Step 4: Using a partition of unity to extend local to global.

Manifolds

Theorem. Let $W \subset \mathbf{R}^n$ be a linear subspace of dimension k. Then there is an orthogonal basis of $\mathbf{R}^n$ in which the first k vectors are a basis for W.

Theorem. There is an isometry (orthogonal transformation) from any k-dimensional linear subspace of dimension k to $\mathbf{R}^k \times \mathbf{0}^{n-k}$.

Theorem. There is a unique function, V, that assigns to every k-tuple of elements in $\mathbf{R}^n$ a non-negative number such that (1) If h: $\mathbf{R}^n \to \mathbf{R}^n$ is an isometry then $V(h(\mathbf{x}_1), \ldots, h(\mathbf{x}_n)) = V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, and (2) If $\mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbf{R}^k \times \mathbf{0}^{n-k} \subset \mathbf{R}^n$, so that $\mathbf{y}_i = [\mathbf{z}_i\ \mathbf{0}]$ then $V(\mathbf{y}_1, \ldots, \mathbf{y}_k) = |\det [\mathbf{z}_1 \ldots \mathbf{z}_n]|$. In addition, $V(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ if and only if $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ are dependent. Notice that V is defined by $V = (\det X^T X)^{1/2}$ where $X = [\mathbf{x}_1 \ldots \mathbf{x}_k]$. We call this <u>volume</u>.

Definition. Let $(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in (\mathbf{R}^n)^k$, $k \le n$. Let $X = [\mathbf{x}_1 \ldots \mathbf{x}_k]$, $I = (i_1, \ldots, i_k)$ be an ascending k-tuple of integers, $1 \le i_1 < \ldots < i_k \le n$. Let X_I be a $k \times k$ matrix with the first row the i_1^{st} row of X, …, the k^{th} row the i_k^{th} row of X.

Theorem. Let X be an $n \times k$ matrix, $k \le n$. Then, $V(X) = (\sum_{[I]} (\det X_I)^2)^{1/2}$ where [I] is the set of all ascending k-tuples from $\{1, 2, \ldots, n\}$.

Definition. Let $k \le n$, A be open in $\mathbf{R}^k$, $\alpha: A \to \mathbf{R}^n$ of class C^r. Then $Y = \alpha(A)$ is a k-dimensional <u>parameterized manifold</u>.

Definition. The volume of a parameterized manifold is $v(Y_\alpha) = \int_A V(D\alpha)$.

Definition. Let f: $Y_\alpha \to \mathbf{R}$. We define the integral of f over Y_α as $\int_{Y\alpha} f\, dV = \int_A (f \circ \alpha)\, V(D\alpha)$.

Theorem. Let g: $A \to B$ be a diffeomorphism of open sets in $\mathbf{R}^k$. Let $\beta: B \to \mathbf{R}^n$ be C^r and $Y = \beta(B)$. Let $\alpha = \beta \circ g$. Then, $\alpha: A \to \mathbf{R}^n$ with $Y = \alpha(A)$. Then, $\int_{Y\alpha} f\, dV = \int_{Y\beta} f\, dV$.

Proof. Apply the Chain Rule and the Change of Variables Theorem.

Definition. Let $k > 0$. Let $M \subset \mathbf{R}^n$. Suppose that for any $\mathbf{p} \in M$ there exists V containing $\mathbf{p}$ such that V is open in M and there exists an open set $U \subset \mathbf{R}^k$ and a one-to-one and onto $\alpha: U \to V$ such that (1) α is C^r, (2) $\alpha^{-1}: V \to U$ is continuous, and (3) $D\alpha$ has rank k everywhere on U. Then we call α a <u>coordinate patch</u> and M is a <u>k-manifold without boundary</u>.

Note. Manifolds are defined locally. The second condition ensures that manifolds do not cross themselves, and the third condition ensures that there are no singularities.

Definition. Let $S \subset \mathbf{R}^k$, f: $S \to \mathbf{R}^n$. f is <u>C^r on S</u> if f may be extended to a function g: $U \to \mathbf{R}^n$ that is C^r on an open set U containing S.

Lemma. If $S \subset \mathbf{R}^k$, f: $S \to \mathbf{R}^n$, and if for all $\mathbf{x} \in S$ there exists a neighborhood, $U_\mathbf{x}$, around $\mathbf{x}$, and $g_x: U_x \to \mathbf{R}^n$ that is C^r with compact support contained in $S \cap U_x$ then f is C^r.

Definition. $\mathbf{H}^k$ is the <u>upper half-space</u> in $\mathbf{R}^k$. $\mathbf{H}^k = \{\mathbf{x} \in \mathbf{R}^k \mid x_k \ge 0\}$. $\mathbf{H}^k_+ = \{\mathbf{x} \in \mathbf{R}^k \mid x_k > 0\}$.

Theorem. Let U be open in $\mathbf{H}^k$ but not $\mathbf{R}^k$ and $\alpha: U \cap \mathbf{H}^k \to \mathbf{R}$. Let $\beta: U \to \mathbf{R}$ be any extension of α. Then $D\alpha = D\beta$ everywhere on $U \cap \mathbf{H}^k$.

Lemma. If M is a manifold in $\mathbf{R}^n$ and α is a coordinate patch on M then the restriction of α to $\mathbf{H}^k$ is also a manifold.

Theorem. Let M be a k-manifold in $\mathbf{R}^n$ of class C^r. Let $\alpha_0: U_0 \to V_0$, $\alpha_1: U_1 \to V_1$ be coordinate patches on M with $W = V_0 \cap V_1 \ne \emptyset$. Let $W_i = \alpha_i^{-1}(W)$. Then $\alpha_1^{-1} \circ \alpha_0: W_0 \to W_1$ is C^r and $D(\alpha_1^{-1} \circ \alpha_0)$ is non-singular.

Proof. Apply the change of variable theorem.

Definition. Let M be a k-manifold in $\mathbf{R}^n$. Let $\mathbf{p} \in$ M. If there is a coordinate patch, α: U → V on M about $\mathbf{p}$ that is open in $\mathbf{R}^k$ then $\mathbf{p}$ is an interior point. If there is no such coordinate patch, then $\mathbf{p}$ is a boundary point. The set of all boundary points is ∂M.

Lemma. Let M be a k-manifold in $\mathbf{R}^n$, α: U → V a coordinate patch on M about $\mathbf{p}$. If U is open in $\mathbf{R}^k$ then $\mathbf{p}$ is an interior point. If U is open in $\mathbf{H}^k$ and $\mathbf{p} = \alpha(\mathbf{x})$ for some $\mathbf{x} \in \mathbf{H}_+^k$, the $\mathbf{p}$ is an interior point of M. If U is open in $\mathbf{H}^k$ and $\mathbf{p} = \alpha(\mathbf{x})$ from some $\mathbf{x} \in \mathbf{R}^{k-1} \times 0$, then $\mathbf{p}$ is a boundary point.

Proof. In the second case, notice that U $\cap$ $\mathbf{H}^k_+$ is open and therefore may be used as a coordinate patch to show that $\mathbf{p}$ is in interior point. In the third case, if there were any open coordinate patch, β, in $\mathbf{R}^k$, then $\beta^{-1} \circ \alpha$ is invertible and would take open sets to open sets. This leads to a contradiction.

Scalar Functions on Manifolds.

Definition. Let M be a k-manifold in $\mathbf{R}^n$. Let f: M → $\mathbf{R}$. Suppose Support(f) $\subset$ V and α: U → V is a coordinate patch. Them, we define $\int_M f = \int_U (f \circ\alpha)\, V(D\alpha)$.

Note. If M is a 2-manifold in $\mathbf{R}^3$ this corresponds to $\int_M f = \int_U (f \circ\alpha)\, \|\partial\alpha/\partial u \times \partial\alpha/\partial v\|\, du\, dv$.

Note. As with parameterized manifolds, we may use the change of variables theorem to show that the parameterization does not matter.

Definition. Let M be a k-manifold in $\mathbf{R}^n$ and f: M → $\mathbf{R}$. Let *V* be a collection of coordinate patches on M. Choose a partition of unity on $\mathbf{R}^n$ dominated by *V* (by extending each V $\in$ *V* to an open set in $\mathbf{R}^n$). Since M is compact, all but finitely many ϕ_i vanish at any point of M. Then, $\int_M f = \sum \int_M \phi_i f$.

Note. The choice of partition of unity does not change the value of the integral.

Note. To find integrals over manifolds that are not compact, take the limit of compact manifolds whose union is M.

Tensors

Definition. Let V be a vector space. If f: V → $\mathbf{R}$ is linear, f is a linear functional.

Definition. The set of all linear functionals corresponding to some vector space is also a vector space. This is the dual of the vector space.

Note. A basis of the dual is $\{f_1, \ldots, f_n\}$ such that $f_i(b_j) = 1$ if i = j, 0 otherwise, where $\{b_1, \ldots, b_n\}$ is a basis for the vector space.

Note. In a finite dimensional vector space, the dual of the dual is the original vector space.

Definition. f: V^k → $\mathbf{R}$ is a tensor if f is multi-linear (linear in the i^{th} coordinate, when all other coordinates are fixed, for all i). The set of all k-tensors on V is $L^k(V)$.

Theorem. The set of all k-tensors is a vector space.

Lemma. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis of V. If f, g: V^k → $\mathbf{R}$ are k-tensors on V and $f(\mathbf{a}_{i1}, \ldots, \mathbf{a}_{ik}) = g(\mathbf{a}_{i1}, \ldots, \mathbf{a}_{ik})$ for all $I = (i_1, \ldots, i_k)$ chosen from {1, 2, …, n}, then f = g.

Theorem. There is a unique ϕ_I: V^k → $\mathbf{R}$ such that for all $J = (j_1, \ldots, j_k)$, $\phi_I(a_{j1}, \ldots, a_{jk}) = 1$ if I = J, 0 if I $\neq$ J. $\{\phi_I\}$ is a basis for $L^k(V)$.

Definition. Let f be a k-tensor and g an l-tensor over the same vector space. The tensor product is defined by $f \otimes g\, (\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})$.

Theorem. If ϕ_I is a k-tensor with $I = (i_1, \ldots, i_k)$, then $\phi_I = \phi_{i1} \otimes \ldots \otimes \phi_{ik}$, where ϕ_{ij} is a 1-tensor.

Theorem. The tensor product is associative, homogeneous ($(cf) \otimes g = f \otimes (cg) = c(f \otimes g)$), and distributive.
Definition. An elementary permutation is a permutation that switches i and i+1, for some i.
Note. All permutations are the product of elementary permutations.
Definition. A k-tensor on V is alternating if $f(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \mathbf{v}_i, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_n) = - f(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. The set of all alternating k-tensors is $A^k(V)$.
Note. $A^k(V)$ is a subspace of $L^k(V)$.
Theorem. Let V be a vector space with basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let $I = (i_1, \ldots, i_k)$ be an ascending k-tuple. There exists a unique k-tensor, Ψ_I, on V, such that for all ascending k-tuples, J, $\Psi_I(\mathbf{a}_{j1}, \ldots, \mathbf{a}_{jk}) = 1$ if I = J and 0 otherwise. These tensors form a basis for $A^k(V)$.
Note. We may also define $\Psi_I = \sum_\sigma (\text{sgn } \sigma) (\phi_I)^\sigma$, for all $\sigma \in S_n$, where $(\phi_I)^\sigma$ applies ϕ_I to the permutation σ of the vectors.
Note. $\Psi_I(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \det X_I$.
Example. The alternating 2-tensors on $\mathbf{R}^3$ are $\Psi_{(1,2)} = \phi_{(1,2)} - \phi_{(2,1)} = x_1y_2 - x_2y_1$, $\Psi_{(1,3)} = x_1y_3 - x_3y_1$, and $\Psi_{(2,3)} = x_2y_3 - x_3y_2$.
Note. The number of k-tensors on $\mathbf{R}^n$ is n^k. The number of alternating k-tensors on $\mathbf{R}^n$ is $\binom{n}{k}$.
Definition. We define a linear transformation A: $L^k(V) \to L^k(V)$ by $Af = \sum_\sigma (\text{sgn } \sigma) f^\sigma$. If f is an alternating k-tensor and g is an alternating l-tensor on V, we define the wedge product, an alternating k+l tensor on V, by $f \wedge g = A(f \otimes g)/ k!\, l!$.
Note. $\psi_I = A\phi_I$.
Note. For 1-tensors, $\phi_i \wedge \phi_j = \phi_i \otimes \phi_j$.
Theorem. The wedge product is associative, homogeneous, and distributive. In addition, $g \wedge f = (-1)^{kl} f \wedge g$. The wedge product is preserved under the pullback: $T^*(f \wedge g) = (T^*f) \wedge (T^*g)$.
Note. Since $\phi_i \wedge \phi_j = - \phi_j \wedge \phi_i$, $\phi_i \wedge \phi_i = 0$.
Theorem. $\Psi_I = \phi_{i1} \wedge \ldots \wedge \phi_{ik}$.
Definition. Let $\mathbf{x} \in \mathbf{R}^n$. A tangent vector to $\mathbf{R}^n$ at $\mathbf{x}$ is $(\mathbf{x}; \mathbf{v})$, where $\mathbf{v} \in \mathbf{R}^n$. We define: $(\mathbf{x}; \mathbf{v}) + (\mathbf{x}; \mathbf{w}) = (\mathbf{x}; \mathbf{v} + \mathbf{w})$ and $c(\mathbf{x}; \mathbf{v}) = (\mathbf{x}; c\mathbf{v})$. The set of all tangent vectors to $\mathbf{R}^n$ at $\mathbf{x}$ is called the tangent space to $\mathbf{R}^n$ at $\mathbf{x}$, or $T_\mathbf{x}(\mathbf{R}^n)$.
Definition. Let A be open in $\mathbf{R}^k$ or $\mathbf{H}^k$, α: A $\to \mathbf{R}^n$ be of class C^r. Let $\mathbf{x} \in$ A and $\mathbf{p} = \alpha(\mathbf{x})$. We define $\alpha_*: T_\mathbf{x}(\mathbf{R}^k) \to T_\mathbf{p}(\mathbf{R}^n)$ by $\alpha_*(\mathbf{x}; \mathbf{v}) = (\mathbf{p}; D\alpha(\mathbf{x}) \bullet \mathbf{v})$. This is the transformation induced by α and a push-forward.
Definition. Let $(a, b) \subset \mathbf{R}$. Let γ: $(a, b) \to \mathbf{R}^n$ be C^r. We define the velocity vector of γ corresponding to t to be $(\gamma(t); D\gamma(t))$.
Note. Velocity vectors are special cases of push-forwards.
Lemma. Let A be open in $\mathbf{R}^k$ or $\mathbf{H}^k$. Let α: A $\to \mathbf{R}^m$ be C^r. Let B be open in $\mathbf{R}^m$ or $\mathbf{H}^m$, with $\alpha(A) \subset B$. Let β: B $\to \mathbf{R}^n$ be C^r. Then, $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.
Definition. Let M be a k-manifold of class C^r in $\mathbf{R}^n$. If $\mathbf{p} \in$ M, choose a coordinate patch α: U $\to$ V about $\mathbf{p}$. Let $\mathbf{x} \in$ U such that $\alpha(\mathbf{x}) = \mathbf{p}$. Then we define the tangent space to M at $\mathbf{p}$ by $T_\mathbf{p}(M) = \alpha_*(T_\mathbf{x}(\mathbf{R}^k)) = \{\alpha_*(\mathbf{x}; \mathbf{v}) \mid \mathbf{v} \in \mathbf{R}^k\}$.
Note. $T_\mathbf{p}(M)$ is a linear subspace of $T_\mathbf{x}(\mathbf{R}^n)$.
Definition. The union of $T_\mathbf{p}(M)$ over all $\mathbf{p} \in$ M is called the tangent bundle of M.

Definition. Let $A \subset \mathbf{R}^n$ be open. A <u>tangent vector field</u> in A is a continuous function F: $A \to \mathbf{R}^n \times \mathbf{R}^n$ such that $F(\mathbf{x}) \in T_{\mathbf{x}}(\mathbf{R}^n)$. Thus, we may write $F(\mathbf{x}) = (\mathbf{x}; f(\mathbf{x}))$ where f: $A \to \mathbf{R}^n$. If F is of class C^r we say the tangent vector field is of class C^r. The tangent vector field to a manifold is F: $M \to T(M)$.

Definition. Let $A \subset \mathbf{R}^n$ be open. A <u>k-tensor field</u>. in A is ω: $\mathbf{x} \to L^k(T_{\mathbf{x}}(\mathbf{R}^n))$; in other words, ω assigns a k-tensor defined on $T_{\mathbf{x}}(\mathbf{R}^n)$ to each $\mathbf{x} \in A$. Notice that $\omega(\mathbf{x})((\mathbf{x}; \mathbf{v}_1), \ldots, (\mathbf{x}; \mathbf{v}_k))$ must be continuous as a function of $\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_k$. If $\omega(\mathbf{x})$ is an alternating k-tensor for all $\mathbf{x}$, we call ω a <u>differential form of order k</u> on A.

Definition. The <u>elementary 1-forms</u> on $\mathbf{R}^n$ are given by $\phi_i(\mathbf{x})(\mathbf{x}; \mathbf{e}_j) = 1$ if $i = j$, 0 otherwise. The <u>elementary k-forms</u> on $\mathbf{R}^n$ are given by $\psi_I(\mathbf{x}) = \phi_{i1}(\mathbf{x}) \wedge \ldots \wedge \phi_{ik}(\mathbf{x})$, where $I = (i_1, \ldots, i_k)$ is an ascending k-tuple.

Note. If ω is a k-form on A, we may write $\omega(\mathbf{x}) = \sum_{[I]} b_I(\mathbf{x})\, \psi_I(\mathbf{x})$, where the b_I are scalar functions and are called the components of ω.

Lemma. Let ω be a k-form on A open in $\mathbf{R}^n$. The ω is C^r if and only if all its components are C^r.

Lemma. Let ω, η be k-forms and θ be an l-form on A. If they are all C^r, so are $a\omega + b\eta$ and $\eta \wedge \theta$.

Definition. Let A be open in $\mathbf{R}^n$. If f: $A \to \mathbf{R}$ is C^r, f is called a <u>scalar field</u> in A and a <u>differential form of order 0</u>.

Note. $\omega(\mathbf{x}) \wedge f(\mathbf{x}) = f(\mathbf{x})\, \omega(\mathbf{x})$.

Definition. Let A be open in $\mathbf{R}^n$ and f: $A \to \mathbf{R}$ be C^∞. The, $d(f(\mathbf{x}; \mathbf{v})) = Df(\mathbf{x})\bullet\mathbf{v}$. We call this the <u>differential</u> of f.

Lemma. Let π_i: $\mathbf{R}^n \to \mathbf{R}$, where $\pi_i(\mathbf{x}) = x_i$ (the i^{th} projection function). Then, $d\pi_i = \phi_i$. (We generally write this as dx_i.)

Theorem. Let A be open in $\mathbf{R}^n$ and f: $A \to \mathbf{R}$ be C^∞. Then, $df = (D_1f)dx_1 + \ldots + (D_nf)dx_n$.

Theorem. d is linear on 0-forms.

Note. $dx_I = dx_{i1} \wedge \ldots \wedge dx_{ik} = \psi_I$. $dx_I(\mathbf{x})((\mathbf{x}; \mathbf{v}_1), \ldots, (\mathbf{x}, \mathbf{v}_k)) = \det V_I$.

Definition. We define $\Omega^k(A)$ to be the set of all k-forms on A.

Definition. Let $\omega \in \Omega^k(A)$, $j > 0$. Let $\omega = \sum_{[I]} f_I\, dx_I$. We define $d\omega = \sum_{[I]} df_I \wedge dx_I$.

Theorem. Let d: $\Omega^k(A) \to \Omega^{k+1}(A)$.

- d is linear ($d(a\omega + b\eta) = a(d\omega) + b(d\eta)$.).
- $df(\mathbf{x})(\mathbf{x}; \mathbf{v}) = Df(\mathbf{x})\mathbf{v}$ if f is a 0-form.
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\, \omega \wedge d\eta$, where ω is a k-form and η is any form.
- $d(d\omega) = 0$ for all forms.

Definition. A form, ω, is <u>closed</u> if $d\omega = 0$.

Definition. A form, ω, is <u>exact</u> if $\omega = d\theta$ for some k-1 form θ.

Proposition. Every exact form is closed.

Definition. Let A be open in $\mathbf{R}^n$. Let f: $A \to \mathbf{R}$ be a scalar field. We define the <u>gradient</u> of f by $(\text{grad } f)(\mathbf{x}) = (\mathbf{x}; D_1f(\mathbf{x})\mathbf{e}_1 + \ldots + D_nf(\mathbf{x})\mathbf{e}_n)$. Let $G(\mathbf{x}) = (\mathbf{x}; g(\mathbf{x}))$ be a vector field on A, with $g(\mathbf{x}) = g_1(\mathbf{x})\mathbf{e}_1 + \ldots + g_n(\mathbf{x})\mathbf{e}_n$. We define the <u>divergence</u> of G by $(\text{div } G) = D_1g_1(\mathbf{x}) + \ldots + D_ng_n(\mathbf{x})$.

Theorem. Let A be open in $\mathbf{R}^n$. Then we have the following vector space isomorphisms:

α_0: Scalar fields in $A \to \Omega^0(A)$.

α_1: Vector fields in $A \to \Omega^1(A)$

β_{n-1}: Vector fields in A → $\Omega^{n-1}(A)$

β_n: Scalar field in A → $\Omega^n(A)$

so that $d \circ \alpha_0 = \alpha_1 \circ$ grad and $d \circ \beta_{n-1} = \beta_n \circ$ div.

Proof. $\alpha_0(f) = f$

$\alpha_1(F) = \sum f_i \, dx_i$

$\beta_{n-1}(G) = \sum (-1)^{i-1} g_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots dx_n$

$\beta_n(h) = h \, dx_1 \wedge \ldots \wedge dx_n$

Definition. Let $A \subset \mathbf{R}^3$ be open. Let $F(\mathbf{x}) = (\mathbf{x}; \sum f_i(\mathbf{x})\mathbf{e}_i)$ be a vector field in A. We define the vector field, curl F, by $(\text{curl } F)(\mathbf{x}) = (\mathbf{x}; (D_2f_3 - D_3f_2)(\mathbf{x})\mathbf{e}_1 + (D_3f_1 - D_1f_3)(\mathbf{x})\mathbf{e}_2 + (D_1f_2 - D_2f_1)(\mathbf{x})\mathbf{e}_3)$.

Theorem. Let A be open in $\mathbf{R}^3$. Then, in addition to the isomorphisms in the previous theorem, we find that $d \circ \alpha_1 = \beta_2 \circ$ curl.

Corollary. Since $d(d\omega) = 0$, curl(grad f) = 0 and div(curl F) = 0.

Definition. Let B be open in $\mathbf{R}^n$ and $\alpha(A) \subset B$. A <u>dual transformation of forms</u> (<u>pullback</u>) is given by $(\alpha^*f)(\mathbf{x}) = f(\alpha(\mathbf{x}))$ if f is a 0 form, and $((\alpha^*\omega)(\mathbf{x}))(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \omega(\alpha(\mathbf{x}))(\alpha_*(\mathbf{x}; \mathbf{v}_1), \ldots, \alpha_*(\mathbf{x}; \mathbf{v}_n))$.

Note. $\omega(\mathbf{y}) \in A^k(T_{\mathbf{y}}(\mathbf{R}^n))$ and $T^*(\omega(\mathbf{y})) = (\alpha^*\omega)(\mathbf{x})$.

Proposition. Let ω, η, and θ be forms, with ω and η having the same order. Then:

- $\alpha^*(a\omega + b\eta) = a\,\alpha^*(\omega) + b\,\alpha^*(\eta)$ [linear]
- $\alpha^*(\omega \wedge \theta) = \alpha^*(\omega) \wedge \alpha^*(\theta)$
- $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

Theorem. Let A be open in $\mathbf{R}^k$. Let α: A → $\mathbf{R}^n$ be C^∞. Let $\mathbf{x} \in \mathbf{R}^k$ and $\mathbf{y} \in \mathbf{R}^n$ with $\alpha(\mathbf{x}) = \mathbf{y}$. If $I = (i_1, \ldots, i_l)$ is an ascending l-tuple from {1, 2, …, n} then $\alpha^*(dy_I) = \sum_{[J]} \det (\partial\alpha_I / \partial x_J) \, dx_J$.

Proof. $\alpha^*(dy_I) = \sum_{[J]} b_J \, dx_J$ for some scalar functions b_J. For any specific J, $b_J(\mathbf{x}) = \alpha^*(dy_I)(\mathbf{x})((\mathbf{x}; \mathbf{e}_{j1}), \ldots, (\mathbf{x}; \mathbf{e}_{jl})) = dy_I(\mathbf{y})((\mathbf{y}; \partial\alpha(\mathbf{x})/\partial x_{j1}), \ldots, (\mathbf{y}; \partial\alpha(\mathbf{x})/\partial x_{jl})) = \det [\partial\alpha_I/\partial x_J]_I$.

Theorem. Let ω be an l-form defined on an open set containing $\alpha(A)$. Then, $\alpha^*(d\omega) = d(\alpha^*\omega)$.

Proof. Use the chain rule to prove for 0-forms. By linearity, only $\omega = f \, dy_I$ (for one I) needs to be proven.

Integrating over a Parameterized Manifold.

Definition. Let A be open in $\mathbf{R}^k$. Let α: A → $\mathbf{R}^n$ be C^∞, so that $Y = \alpha(A)$ is a parameterized manifold. If ω is a k-form defined on an open set B, $Y \subset B$, then we define $\int_Y \omega = \int_A \alpha^*\omega$ if the latter integral exists.

Note. $\int_A f \, dx_1 \wedge .. \, dx_n = \int_A f$, for $A \subset \mathbf{R}^n$.

Theorem. Let g: A → B be a diffeomorphism of sets in $\mathbf{R}^k$. Assume that det Dg is of constant sign on A. Let β: B → $\mathbf{R}^n$ be C^∞. Let $Y = \beta(B)$. Let $\alpha = \beta \circ g$, so that α: $\mathbf{R}^k$ → $\mathbf{R}^n$ and $Y = \alpha(A)$. If ω is a k-form on $\mathbf{R}^n$ defined on an open set containing Y, the ω is integrable over Y_β if and only if it is integrable over Y_α. Also, $\int_{Y\alpha} \omega = \pm \int_{Y\beta} \omega$, where the sign corresponds to the sign of det Dg.

Proof. Use the change of variables theorem to show $\int_A (\beta \circ g)^*\omega = \int_A \beta^*\omega$.

Theorem. Let $\omega = f \, dz_I$. Then, $\int_{Y\alpha} \omega = \int_A \alpha^*\omega = \int_A (f \circ \alpha) \det (\partial\alpha_I/\partial x)$.

Definition. A frame, $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, with $\mathbf{a}_i \in \mathbf{R}^n$ is <u>right-handed</u> if $\det [\mathbf{a}_1 \ldots \mathbf{a}_n] > 0$.

Definition. Let g: A → B be a diffeomorphism of open sets in $\mathbf{R}^n$. g is orientation-preserving if det Dg is right-handed.

Note. If A is connected and g is a diffeomorphism, then det Dg has constant sign.

Definition. Let M be a k-manifold in $\mathbf{R}^n$. Let α_0: $U_0 \to V_0$ and α_1: $U_1 \to V_1$ be coordinate patches on M. If $V_0 \cap V_1$ is non-empty, then α_0 and α_1 overlap. If α_0 and α_1 overlap and $\alpha_1^{-1} \circ \alpha_0$ is orientation-preserving, then α_0 and α_1 overlap positively. If α_0 and α_1 overlap and $\alpha_1^{-1} \circ \alpha_0$ is not orientation-preserving, then α_0 and α_1 overlap negatively.

Definition. If we may cover M by coordinate patches that overlap positively or not at all, we call M orientable.

Definition. The collection of coordinate patches that cover M and overlap positively is called an orientation of M. M, together with an orientation, is called an oriented manifold.

Definition. Let M be a compact oriented k-manifold in $\mathbf{R}^n$. Let ω be a k-form defined in an open set of $\mathbf{R}^n$ containing M. Let C = M ∩ Support ω; note that C is compact. Suppose there is a coordinate patch α: U → V on M belonging to the orientation of M with $C \subset V$. Assume U is bounded. We define the integral of ω over M as $\int_M \omega = \int_{\text{Int } U} \alpha^*(\omega)$.

Definition. Let M be a compact oriented k-manifold in $\mathbf{R}^n$. Let ω be a k-form defined in an open set of $\mathbf{R}^n$ containing M. Cover M by coordinate patches belonging to the orientation of M; choose a partition of unity on M dominated by these coordinate patches. We define $\int_M \omega = \sum (\int_M \phi_i\, \omega)$.

Definition. Let M be a 1-manifold. The unit tangent vector at $\mathbf{p} \in M$ if given by $T(\mathbf{p}) = (\mathbf{p}; D\alpha(t)/\|D\alpha(t_0)\|)$ where $\alpha(t_0) = \mathbf{p}$.

Definition. So that we may have outward-pointing unit tangent vectors, we define the left-half-line, $\mathbf{L} = \{x \mid x \le 0\}$ and allow coordinate patches α: $\mathbf{L} \to \mathbf{R}^n$.

Definition. Let M be an n-1 manifold in $\mathbf{R}^n$. Let $\mathbf{p} \in M$. Let $(\mathbf{p}; \mathbf{n})$ be a tangent vector to $\mathbf{R}^n$ that is orthogonal to the tangent space to M at **p**. Let $\|\mathbf{n}\| = 1$. If **n** is always pointing the "same" direction, this is called a normal vector field to M and defines an orientation.

Definition. Let M be an n-manifold in $\mathbf{R}^n$. The natural orientation of M is the set of all coordinate patches α: $\mathbf{R}^n \to \mathbf{R}^n$ with det Dα > 0.

Theorem. Let k > 1. If M is an oriented k-manifold in $\mathbf{R}^n$ with ∂M non-empty, then ∂M is orientable.

Proof. Define $b(x_1, \dots, x_{k-1} = (x_1, \dots, x_{k-1}, 0)$. Then the restricted patches for ∂M are $\alpha \circ b$. These define an orientation.

Definition. Let M be an orientable k-manifold in $\mathbf{R}^n$, with ∂M non-empty. Given an orientation of M, the induced orientation of ∂M is defined by the orientation of the restricted coordinate patch if k is even and the opposite orientation if k is odd.

Note. The induced orientation of an n-1 manifold that is the boundary of a naturally oriented n-manifold always points outward from the manifold.

Lemma. Let η be a k-1 form in $\mathbf{R}^k$ defined on an open set containing $I^k = [0, 1]^k$, where η vanishes on Bd I^k, except possibly on $\mathbf{R}^{k-1} \times 0$ (the bottom face). Let b: $\mathbf{R}^{k-1} \to \mathbf{R}^k$ be given by $b(\mathbf{x}) = (\mathbf{x}, 0)$. Then, $\int_{Ik} d\eta = (-1)^k \int_{\text{Int } I(k-1)} b^*\eta$.

Proof. By linearity, let $\eta = f\, dx_{I_j}$, where $I_j = (1, \ldots, j-1, j+1, \ldots, k)$ so $d\eta = (-1)^{j-1} D_j f\, fx_I$, where $I = (1, \ldots, k)$. Then,

$$\int_{\text{Int } I(k)} d\eta = (-1)^{j-1} \int_{I(k-1)} \int_{[0,1]} D_j f(x_1, \ldots, x_k)$$
$$= (-1)^{j-1} \int_{I(k-1)} f(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_k) - f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_k)$$
$$= (-1)^{k-1} \int_{I(k-1)} - f(x_1, \ldots, x_{k-1}, 0)$$

[if j = k since f vanishes on other boundaries, 0 otherwise]

$$= (-1)^k \int_{I(k-1)} f \circ b \text{ or } 0.$$

Since $b^*(dx_{I_j}) = \det (Db)_{I_j}\, dx_1 \wedge \ldots \wedge dx_{k-1} = dx_1 \wedge \ldots \wedge dx_{k-1}$ if j=k, 0 otherwise, $\int_{\text{Int } I(k-1)} b^*\eta = \int_{\text{Int } I(k-1)} f \circ b$ if j=k, 0 otherwise $= (-1)^k \int_{\text{Int } I(k)} d\eta$.

Stokes Theorem. Let M be an oriented k manifold in $\mathbf{R}^n$. Let ∂M have the induced orientation. Let ω be a k-1 form on an open set containing M. Then, $\int_M d\omega = \int_{\partial M} \omega$.

Proof. (For k > 1.) Choose coordinate patches contained in I^k, such that boundary points are in $(\text{Int } I^{k-1}) \times 0$. Use partitions of unity and linearity.

Definition. A <u>0-manifold</u> is a finite collection of points, $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ in $\mathbf{R}^n$. We define an orientation on such a manifold by a function $\varepsilon: \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \to \{-1, 1\}$. $\int_M f = \sum \varepsilon(\mathbf{x}_i) f(\mathbf{x}_i)$.

Definition. Let M be an oriented 1-manifold in $\mathbf{R}^n$. We define the orientation of ∂M by $\varepsilon(\mathbf{p}) = -1$ if there exists $\alpha: U \to V$, about $\mathbf{p}$, with $U \subset \mathbf{H}^k$ and $\varepsilon(\mathbf{p}) = 1$ otherwise.

Note. With this definition, Stokes' theorem holds in this case as well.

Classical Stokes' Theorem. Let F be a vector field in $\mathbf{R}^3$. Then, $\iint_S \nabla \times \mathbf{F}\, d\mathbf{A} = \int_{\partial S} \mathbf{F}\, d\mathbf{S}$.

Theorem. Let M be a compact, oriented n-1 manifold in $\mathbf{R}^n$. Let N be the unit normal field (corresponding to the induced orientation). Let G be a vector field on an open set containing M, so that $G(\mathbf{y}) = (\mathbf{y}; g(\mathbf{y})) = (\mathbf{y}; \sum g_i(\mathbf{y})\, \mathbf{e}_i)$. Let $\omega = \sum (-1)^{i-1} g_i\, dy_1 \wedge \ldots \wedge dy_{i-1} \wedge dy_{i+1} \wedge \ldots \wedge dy_n$. Then, $\int_M \omega = \int_M \langle G, N \rangle\, dV$.

Theorem. Let M be an n-manifold in $\mathbf{R}^n$. Let $\omega = h\, dx_1 \wedge \ldots \wedge dx_n$. Then, $\int_M \omega = \int_M h\, dV$.

Divergence Theorem. Let M be a compact, oriented n-manifold in $\mathbf{R}^n$. Let **N** be the unit normal field. If G is a vector field, then $\int_M (\text{div } G)\, dV = \int_{\partial M} \langle G, \mathbf{N} \rangle\, dV$.

Classical Stokes Theorem. Let M be a compact, oriented 2-manifold in $\mathbf{R}^3$. Let **N** be the unit normal field. Let **F** be a C^∞ function. Then, if $\partial M = \emptyset$, then $\int_M \langle \text{curl } F, N \rangle\, dV = 0$. Otherwise, $\int_M \langle \text{curl } F, N \rangle\, dV = \int_{\partial M} \langle F, T \rangle\, dV$, where T is the unit tangent field to ∂M with the induced orientation.

Lebesgue Measure and the Lebesgue Integral

Definition. A <u>σ-algebra</u>, or a <u>Borel field</u>, is an algebra of sets that is closed under countable union (and therefore countable intersection).

Definition. A <u>Borel set</u> is the smallest σ-algebra that contains the closed and open intervals.

Definition. The <u>outer measure</u> of a set $E \subset \mathbf{R}$ is $m^*E = \inf_{E \subset \cup I(n)} \sum l(I_n)$ (where l(I) is the length of the interval), so that $\{I_n\}$ is a set of intervals that covers E.

Proposition. The outer measure of an interval is its length.

Proposition. Let $\{A_n\}$ be any countable collection of sets. Then, $m^*(\cup A_n) \leq \sum m^*A_n$. (This is called <u>countable subadditivity</u>.)

Proof. Use the $\varepsilon/2^n$ trick.

Corollary. If A is a countable set, then $m^*A = 0$.

Corollary. [0, 1] is uncountable.
Proposition. Given any set A and $\varepsilon > 0$, there exists an open set O such that $A \subset O$ and $m^*O \le m^*A + \varepsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.
Proof. Since m* is defined as an infimum, we may choose $O = \cup I_n$ whose measure is at most ε more that m*A. Take the countable intersection of the O corresponding to 1/n for each n to make G.
Definition. E is <u>measurable</u> if, for all A, $m^*A = m^*(A \cap E) + m^*(A \cap E^C)$.
Note. If E is measurable, so is E^C. $\varnothing$ and **R** are measurable.
Note. By countable subadditivity, $m^*A \le m^*(A \cap E) + m^*(A \cap E^C)$. Thus, we need only prove the other direction to show that something is measurable.
Lemma. If $m^*E = 0$, E is measurable.
Lemma. If E_1 and E_2 are measurable, so are $E_1 \cup E_2$ and $E_1 \cap E_2$.
Corollary. The measurable sets are an algebra (closed under complement, intersection, and union.)
Lemma. Let A be any set. Let $E_1, \ldots, E_n$ be a finite sequence of disjoint measurable sets. Then, $m^*(A \cap (\cup E_i)) = \sum m^*(A \cap E_i)$.
Theorem. The set of measurable sets is a σ-algebra (since it is closed under countable union as well).
Lemma. (a, ∞) is measurable.
Theorem. Every Borel set is measurable.
Proof. All the open sets are measurable and the measurable sets form an algebra.
Proposition. Let $\{E_i\}$ be an infinite, decreasing (ie. $E_{i+1} \subset E_i$) sequence of measurable sets. Let $mE_1 < \infty$. Then, $m(\cap E_i) = \lim mE_n$.
Proof. Construct a sequence of $F_i = E_i - E_{i+1}$.
Proposition. Let E be a set. The following are equivalent:

i. E is measurable.
ii. Given $\varepsilon > 0$ there exists an open set $O \supset E$ with $m^*(O - E) < \varepsilon$.
iii. Given $\varepsilon > 0$ there exists a closed set $F \subset E$ with $m^*(E - F) < \varepsilon$.
iv. There is a $G \in G_\delta$ with $E \subset G$ such that $m^*(G - E) = 0$.
v. There is an $F \in F_\sigma$ with $F \subset E$ such that $m^*(E - F) = 0$.
vi. (If $m^*E < \infty$, then) If $\varepsilon > 0$ there exists a *finite* union $U = \cup I_i$ such that $m^*((U - E) \cup (E - U)) < \varepsilon$.

Proof. (i → ii → vi → ii in the finite case, then i → ii → iv → i, then i → iii → v → i.)
Proposition. The measure of a set is translation invariant.
Proposition. Let f be any function. Let $a \in$ **R**. The following are equivalent:

i. $\{x \mid f(x) > a\}$ is measurable.
ii. $\{x \mid f(x) < a\}$ is measurable.
iii. $\{x \mid f(x) \le a\}$ is measurable.
iv. $\{x \mid f(x) \ge a\}$ is measurable.

Definition. A function is <u>measurable</u> if its domain is measurable and the conditions above hold.
Proposition. Let $c \in$ **R** and f, g be measurable functions on the same domain. Then, f + c, cf, f + g, and fg are measurable.

Theorem. Let $\{f_n\}$ be a sequence of measurable functions defined on the same domain. Then the functions $\sup\{f_1, \ldots, f_n\}$, $\inf\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\limsup f_n$, and $\liminf f_n$ are also measurable.

Theorem. If f is measurable and f = g almost everywhere, then g is measurable.

Littlewood's Three Principles. Every measurable set is nearly a union of intervals. Every measurable function is nearly continuous. Every convergent sequence of measurable functions is nearly uniformly convergent.

Definition. A <u>simple function</u>, φ, is defined by $\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$, where χ_{E_i} is the <u>characteristic function</u> of E_i (1 on E_i, 0 elsewhere).

Note. φ is simple if and only if it is measurable and takes on a finite number of values.

Definition. The canonical representation of φ is $\varphi(x) = \sum a_i \chi_{E_i}(x)$ and $E_i = \{x \mid \varphi(x) = a_i\}$.

Definition. Let $\varphi = \sum a_i \chi_{E_i}$. We define $\int \varphi = \sum a_i mE_i$.

Proposition. Let f be a bounded function on a set E of finite measure. $\inf_{\psi \geq f} \int_E \psi = \sup_{\varphi \leq f} \int_E \varphi$ if and only if f is measurable.

Proof. Suppose $|f| \leq M$. For a fixed n, let $E_k = \{x \mid kM/n \geq f(x) \geq (k-1)M/n\}$. Choose step functions $\psi_n = (M/n) \sum k\chi_{E_k}(x)$ and $\varphi_n = (M/n) \sum (k-1)\chi_{E_k}(x)$. With the proper choice of n, the difference in their integrals can be as small as we want. Conversely, choose ψ_n and φ_n so that $\int_E \psi_n - \int_E \varphi_n < 1/n$. We show $m\{x \mid \varphi_n(x) < \psi_n(x) - 1/v\} < v/n$. We may choose ψ^* and φ^* as the sup and inf of these sequences. They must agree with f almost everywhere. So f is measurable.

Definition. If f is bounded and measurable on a set E of finite measure, we define $\int_E f = \inf_{\psi \geq f} \int_E \psi = \sup_{\varphi \leq f} \int_E \varphi$.

Proposition. If f and g are bounded, measurable functions defined on a set E of finite measure, then:

- $\int_E (af + bg) = a\int_E f + b\int_E g$.
- If g = f almost everywhere then $\int_E g = \int_E f$.
- If $g \leq f$ almost everywhere, then $\int_E g \leq \int_E f$.
- If $A \leq f(x) \leq B$ almost everywhere, then $A(mE) \leq \int_E f \leq B(mE)$.
- If $A \cap B = \varnothing$ and A, B have finite measure, $\int_{A \cup B} f = \int_A f + \int_B f$.

Definition. If f is non-negative and measurable on any measurable set E, we define $\int_E f = \sup_{h \leq f} \int_E h$ where h is bounded, measurable, and non-zero only on a set of finite measure.

Definition. f is <u>integrable</u> over the measurable set E if $\int_E f < \infty$.

Definition. Let f be any function. Define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\} = \max\{-f(x), 0\}$.

Note. $f = f^+ - f^-$. $|f| = f^+ + f^-$.

Definition. Let f be measurable. f is <u>integrable</u> over E if f^+ and f^- are integrable. Then, $\int_E f = \int_E f^+ - \int_E f^-$.

Proposition. The following properties hold for the general Lebesgue integral:

- $\int_E (af + bg) = a\int_E f + b\int_E g$.
- If g = f almost everywhere then $\int_E g = \int_E f$.
- If $g \leq f$ almost everywhere, then $\int_E g \leq \int_E f$.
- If $A \cap B = \varnothing$ and A, B have finite measure, $\int_{A \cup B} f = \int_A f + \int_B f$.

Bounded Convergence Theorem. Let $<f_n>$ be a sequence of measurable functions defined on a set E of finite measure. Suppose there is some real number M such that $|f_n(x)| \leq M$ for all n, x. If $f(x) = \lim f_n(x)$ almost everywhere in E, then $\int_E f = \lim \int_E f_n$.

Proof. Given $\varepsilon > 0$ there exists N and a measurable set $A \subset E$ such that $mA < \varepsilon/4M$ and for all $n > N$ and $x \in E - A$, $|f_n(x) - f(x)| < \varepsilon / 2mE$. Then, $|\int_E f_n - \int_E f| \leq \int_E |f_n - f| = \int_{E-A} |f_n - f| + \int_A |f_n - f| < \varepsilon$.

Fatou's Lemma. Let $<f_n>$ be a sequence of non-negative, measurable functions with $\lim f_n(x) = f(x)$ almost everywhere on a measurable set E. Then, $\int_E f \leq \underline{\lim} \int_E f_n$.

Proof. Let h be a non-negative, bounded, measurable function with $h(x) \leq f(x)$ and $h(x) = 0$ outside a measurable set $E' \subset E$ of finite measure. Let $h_n(x) = \min\{h(x). f_n(x)\}$. Then, h_n is bounded, vanishes outside E', and converges everywhere to h. So we can apply the bounded convergence theorem to find $\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \underline{\lim} \int_E f_n$, for all $h \leq f$. So $\int_E f = \sup \int_E h \leq \underline{\lim} \int_E f_n$.

Monotone Convergence Theorem. Let $<f_n>$ be an increasing sequence of non-negative, measurable functions with $f(x) = \lim f_n(x)$ almost everywhere. Then, $\int f = \lim \int f_n$.

Proof. We know $\int f \leq \underline{\lim} \int f_n$. Since $f_n \leq f$ for all n, $\limsup \int f_n \leq \int f$. So $\int f = \lim \int f_n$.

Lebesgue Convergence Theorem. Let g be integrable over E and $<f_n>$ a sequence of measurable functions with $|f_n| \leq g$ everywhere on E. Let $\lim f_n(x) = f(x)$ almost everywhere on E. Then, $\int_E f = \lim \int_E f_n$.

Proof. Apply Fatou's Lemma to $<g - f_n>$ and $<g + f_n>$ to show that $\int_E f \geq \limsup \int_E f_n$ and $\int_E f \leq \underline{\lim} \int_E f_n$. So, $\int_E f = \lim \int_E f_n$.

Theorem. Let $<g_n>$ be a sequence of measurable functions over E that converge almost everywhere to an integrable function g. Let $<f_n>$ be a sequence of measurable functions with $|f_n| \leq g_n$ and $\lim f_n(x) = f(x)$ almost everywhere. If $\int_E g = \lim \int_E g_n$ then $\int_E f = \lim \int_E f_n$.

Corollary. Let $\{u_n\}$ be a sequence of non-negative measurable functions with $f = \sum u_n$. Then, $\int_E f = \sum \int_E u_n$.

Definition. A sequence $<f_n>$ <u>converges</u> to f <u>in measure</u> if, given $\varepsilon > 0$, there exists N such that, for all $n > N$, $m\{x \mid |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon$.

Example. Consider the sequence of functions with a bump of height 1 that moves across [0, 1] but gets progressively narrower ($1/2^k$ for 2^k consecutive functions). This functions converges in measure, but converges pointwise nowhere.

Proposition. Suppose $<f_n>$ converges to f in measure and all the f_n are measurable. Then there is a subsequence $<f_{nk}>$ that converges to f almost everywhere.

Proof. For each v, choose n_v such that $m\{x: |f_n - f(x)| > 2^{-v}\} < 2^{-v}$. Let $E_v = \{x: |f_n(x) - f(x)| > 2^{-v}\}$. If x is not in $\cup_{v=k}^{\infty} E_v$ then $|f_{nv}(x) - f(x)| < 2^{-v}$ for all $v > k$, and $\lim f_{nv}(x) = f(x)$. Let $A = \cap_k (\cup_{v=k}^{\infty} E_v)$. $\lim f_{nv}(x) = f(x)$ for all x not in A. Since A is an intersection of sets of progressively smaller measure (2^{-v+1} each union), $mA = 0$. So this subsequence converges almost everywhere.

Corollary. Let $<f_n>$ be a sequence of measurable functions defined on a set E of finite measure. Then f_n converges to f in measure if and only if every subsequence of $<f_n>$ has in turn a subsequence that converges to f almost everywhere.

Proposition. The convergence theorems stated above hold if "convergence almost everywhere" is replaced by "convergence in measure."

Proof. (Fatou's Lemma). Suppose f_n converges to f in measure. Then, $\underline{\lim} \int_E f_n = \lim \int_E f_{nk}$ for some subsequence $<f_{nk}>$, that also converges to f in measure. So there is a subsequence $<f_{nkj}>$ that converges to f almost everywhere. So by the previous Fatou's Lemma, $\int_E f \leq \underline{\lim} \int_E f_{nkj} = \lim \int_E f_{nk} = \underline{\lim} \int_E f_n$.
(Monotone Convergence). Suppose $<f_n>$ is an increasing sequence of non-negative functions that converge to f in measure. Note that $x = \lim x_n$ if and only if every subsequence of $<x_n>$ has a subsequence of converges to x. Let $x_n = \int_E f_n$. Every subsequence $<f_{nk}>$ has a subsequence, $<f_{nkj}>$ that converges almost everywhere. So, $\int_E f = \lim \int f_{nkj}$, and $\int_E f = \lim \int_E f_n$.

L^p Spaces

Definition. $f \in L^p$ if $\int_{[0,1]} |f|^p < \infty$.
Definition. $||f||_p = (\int_{[0,1]} |f|^p)^{1/p}$.
Definition. A <u>norm</u>, $|| \ ||$, must satisfy: $||af|| = |a| \, ||f||$, $||f + g|| \leq ||f|| + ||g||$, and $||f|| = 0$ if and only if $f = 0$.
Note. If we consider functions equivalent when they are equal almost everywhere, then L^p is a normed linear space.
Definition. $f \in L^\infty$ if f is bounded almost everywhere and measurable. $||f||_\infty = \text{ess sup } |f(x)| = \inf \{M \mid m\{t \mid f(t) > M\} = 0\}$.
Proposition. $\lim_{p \to \infty} ||f||_p = ||f||_\infty$.
Definition. φ on (a, b) is <u>convex</u> (concave up) if for all $x, y \in (a, b)$ and $\lambda \in [0, 1]$, $\varphi(\lambda x + (1 - \lambda)y) \leq \lambda \, \varphi(x) + (1 - \lambda) \, \varphi(y)$.
Minkowski Inequality. Given $f, g \in L^p$, for $1 \leq p \leq \infty$, $||f + g||_p \leq ||f||_p + ||g||_p$.
Note. If $0 \leq p \leq 1$, then $||f + g||_p \geq ||f||_p + ||g||_p$.
Holder's Inequality. Let $p, q > 0$ and $1/p + 1/q = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $||fg||_1 = \int_{[0,1]} |fg| \leq ||f||_p \, ||g||_p$.
Riesz-Fischer Theorem. L^p is complete for $1 \leq p < \infty$.
Definition. A <u>linear functional</u> on a normed linear space, X, is a mapping $F: X \to \mathbf{R}$ such that $F(af + bg) = a\,F(f) + b\,F(g)$.
Definition. A linear functional is <u>bounded</u> if there exists M such that $|F(f)| \leq M \, ||f||$ for all $f \in X$.
Definition. The <u>norm</u> of a functional, F, is given by $||F|| = \sup_{f \neq 0} |F(f)|/||f||$.
Proposition. If $g \in L^q$ and $1/p + 1/q = 1$, a bounded linear functional on L^p is $F(f) = \int_{[0,1]} fg$.
Riesz Representation Theorem. Let F be any bounded linear functional on L^p, $1 \leq p < \infty$. Then there is a function $g \in L^q$, $1/q + 1/p = 1$, such that, for all $f \in L^p$, $F(f) = \int_{[0,1]} fg$. In addition, $||F|| = ||g||_q$.