Macroeconomics: Heterogeneity

Rebecca Sela

August 13, 2006

1 Heterogeneity in Complete Markets

1.1 Aggregation

Definition An economy admits *aggregation* if the behavior of aggregate quantities and prices does not depend on the distribution of individual choices and states. There is aggregation if we can define a *representative agent* who behaves in equilibrium as the sum of all the individual agents.

When an economy admits aggregation, calculation is generally easier. If every agent makes choices in the same ratio (capital to labor, or consumption to leisure), an economy is likely to admit aggregation.

Proposition 1.1 *In a neoclassical growth model with capital, suppose that every firm has the same technology which has constant returns to scale and is strictly increasing, strictly concave, and differentiable. Then, there is a representative firm.*

Proof Suppose there are $i = 1, ..., m$ firms, each of which produces according to $Zf(k^i, n^i)$, where f has the properties above. Then, the firm's problem is to maximize $Zf(k^i, n^i) - (r + \delta)k^i - wn^i$. This yields the first order conditions for each firm:

$$\begin{aligned} Zf_k(k^i, n^i) &= r + \delta \\ Zf_n(k^i, n^i) &= w \end{aligned}$$

By the linear homogeneity of the constant returns to scale, we may rewrite the conditions in terms of the capital-labor ratio:

$$\begin{aligned} Zf_k\left(\frac{k^i}{n^i}\right) &= r + \delta \\ Zf_n\left(\frac{k^i}{n^i}\right) &= w \\ \frac{f_k\left(\frac{k^i}{n^i}\right)}{f_n\left(\frac{k^i}{n^i}\right)} &= \frac{r + \delta}{w} \end{aligned}$$

By strict concavity, $f_k\left(\frac{k}{n}\right)$ is strictly decreasing and $f_n\left(\frac{k}{n}\right)$ is strictly increasing. Thus, their ratio is decreasing. This means that $\frac{k}{n}$ is uniquely determined by $\frac{r+\delta}{w}$, and all firms must choose the same capital-labor ratio. This means that the technology can be represented by a single firm with production technology, $f(K,N)$, with $K=\sum_{i=1}^{m} k^i$, $N=\sum_{i=1}^{m} n^i$, and $\frac{K}{N}=\frac{k^i}{n^i}$. $\blacksquare$

Proposition 1.2 *In a neoclassical growth model with capital, suppose every consumer has the same endowment and the same strictly concave preferences. Then, there is a formulation of the economy with a representative agent.*

Proof Suppose there are $i=1,...,n$ agents with identical endowments, $k_0^i=\overline{k_0}$, and preferences of the form $\sum_{t=0}^{\infty}\beta^t u(c_t^i)$. If u is strictly concave, then each agent will make exactly the same decision. This will lead to a representative agent. $\blacksquare$

1.2 Neoclassical growth model with heterogeneous endowments of wealth

Suppose there are $i=1,...,N$ types of agents, each with measure μ_i, so that $\sum_{i=1}^{N}\mu_i=1$.

- **Preferences** are given by $\sum_{t=0}^{\infty}\beta^t u(c_t^i)$ with u strictly concave and identical across all types of people. In particular, we assume that utility function is in one of the following classes:

$$\begin{aligned} u(c) &= \gamma\ln(c+\bar{c}),\ \bar{c}+c>0 \\ u(c) &= \gamma(c+\bar{c})^{\sigma},\ \bar{c}+c>0 \\ u(c) &= -\bar{c}\exp(-\eta c) \end{aligned}$$

  where the three utility types are called *log utility*, *power utility*, and *exponential utility*; all lead to homothetic preferences with a *subsistence level of consumption*, $-\bar{c}\geq 0$.

- **Technology** is given by $Y=f(K_t)$, where f is strictly increasing, strictly concave and differentiable (not necessarily CRS). The firm owns the capital and households own shares of the firm. The representative firm solves

$$A_t=\max_{\{I_\tau\}}\sum_{\tau=t}^{\infty}\left(\frac{p_\tau}{p_t}\right)(f(K_\tau)-I_\tau)$$

  subject to $K_{\tau+1}=(1-\delta)K_\tau+I_\tau$. This is the sequence of investment that maximizes the present discounted value of profits, π_t, since $\frac{p_\tau}{p_t}$ is the inverse of the interest rate from time t to τ. A_t is the value of the firm at time t, which is equivalent to the aggregate wealth; each individual is entitled to a proportion of the wealth according to their share in the firm.

- **Markets** are all competitive and complete, with time 0, Arrow-Debreu trading (sequential trading would lead to an equivalent result).

The household problem for a household of type i is to choose $\{c_t\}$ to maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $\sum_{t=0}^{\infty} p_t c_t^i \leq p_0 a_0^i = s_0^i A_0$, where a_0^i is time 0 wealth, which is exogenous. a_0^i varies across $i = 1, ..., N$. For any t, $p_t a_t^i = s_t^i \sum_{\tau=t}^{\infty} p_\tau \pi_\tau$, so that $a_t^i = s_t^i A_t$, where s_t^i is the share of the firm that agent i owns at time t. Using the first order conditions for c_t from the Lagrangian, we find that $\beta^t u'(c_t^i) = \lambda^i p_t$. In particular, for log preferences, $c_t^i = \frac{\beta^t \gamma}{\lambda^i p_t} - \bar{c}$. Substituting this into the lifetime budget constraint, we find that:

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t \left(\frac{\beta^t \gamma}{\lambda^i p_t} - \bar{c} \right) &= p_0 a_0^i \\
\frac{\gamma}{\lambda^i} \left(\sum_{t=0}^{\infty} \beta^t \right) - \bar{c} \sum_{t=0}^{\infty} p_t &= p_0 a_0^i \\
\frac{\gamma}{\lambda^i (1-\beta)} - \bar{c} \sum_{t=0}^{\infty} p_t &= p_0 a_0^i \\
\frac{\gamma}{\lambda^i} &= (1-\beta) p_0 a_0^i + (1-\beta) \bar{c} \sum_{t=0}^{\infty} p_t
\end{aligned}$$

Substituting the last line into the first order condition at time 0 and time t yields:

$$\begin{aligned}
c_0^i &= \frac{\gamma}{\lambda^i p_0} - \bar{c} \\
&= \frac{1}{p_0} \left((1-\beta) p_0 a_0^i + \bar{c}(1-\beta) \sum_{t=0}^{\infty} p_t \right) - \bar{c} \\
&= \bar{c}(1-\beta) \left(\sum_{t=0}^{\infty} \frac{p_t}{p_0} - 1 \right) + (1-\beta) a_0^i \\
c_t^i &= \bar{c}(1-\beta) \left(\sum_{\tau=t}^{\infty} \frac{p_\tau}{p_t} - 1 \right) + (1-\beta) a_t^i
\end{aligned}$$

This shows that the optimal consumption choice depends first on $\theta(p^t, \bar{c}) = \bar{c}(1-\beta)(\sum_{\tau=t}^{\infty} \frac{p_\tau}{p_t} - 1)$, which depends on the future path of prices and on the subsistence level, and then on a term which is linear in wealth. Each of these utility functions leads to linear Engle curves in wealth, where the marginal propensity to save is β. If $\theta > 0$, then the consumption with no wealth will be positive, so that the average propensity to save is increasing in wealth.

In general, to check if aggregation holds, we try to write $c_t^i = \theta a_t^i + f(t)$ where θ does not depend on i and $f(t)$ does not depend on wealth and can also be aggregated. Then, we may aggregate up to $C_t = \theta \sum_i a_t^i + F(t) = \theta A_t + F(t)$.

We may use linearity to find the aggregate dynamics:

$$C_t = \theta(p^t, \bar{c}) + (1-\beta)A_t$$

where no term depends on i. Thus, aggregate consumption does not depend on the distribution of wealth. (Such a result would fail if c_t^i depended on a_t^i in a non-linear way.) Thus, under homothetic preferences, when agents differ only in their initial endowments and there are complete markets, there exists a representative agent representation because of the linearity of the policy function.

The aggregate competitive equilibrium of this model is the maximization of $\sum_{t=0}^{\infty} \beta^t u(C_t)$ subject to $C_t + K_{t+1} \leq F(K_t) + (1-\delta)K_t$ with K_0 given. This model is equivalent to the competitive equilibrium where households own the capital, which in turn is equivalent to the social planner problem. (The social planner problem computes the quantities, which imply the prices.)

In the steady state,

$$\begin{aligned} u'(C^*) &= \beta u'(C^*)(f'(K^*) + 1 - \delta) \\ \frac{1}{\beta} &= f'(K^*) + 1 - \delta \end{aligned}$$

By the agents' first order conditions, $\beta^t u'(c_t^i) = \lambda^i p_t$, so we may compute the interest rate:

$$1 + r^* = \frac{p_t}{p_{t+1}} = \frac{\beta^t u'(C^*)/\lambda^i}{\beta^{t+1} u'(C^*)/\lambda^i} = \frac{1}{\beta}$$

This simplifies the consumption function to

$$c^{*i} = \bar{c}(1-\beta)\left(\sum_{t=\tau}^{\infty} \frac{1}{\beta} - 1\right) + (1-\beta)a_t^i = (1-\beta)a^{*i}$$

so that, in the steady state, agents consume a fixed fraction of their wealth each period; this keeps the wealth distribution constant in the steady state.

The dynamics of the wealth distribution depend on the dynamics of the aggregate variables. At time t,

$$
\begin{aligned}
p_t a_t^i &= p_t c_t^i + \sum_{\tau=t+1}^{\infty} p_\tau c_\tau^i \\
&= p_t c_t^i + p_{t+1} a_{t+1}^i \\
\frac{c_t^i}{a_t^i} &= 1 - \frac{p_{t+1}}{p_t} \\
\frac{a_{t+1}^i}{a_t^i} &= \frac{p_t}{p_{t+1}} \left(1 - \frac{c_t^i}{a_t^i} \right) \\
\frac{c_t^i}{a_t^i} &= \frac{\theta(p_t, \bar{c})}{a_t^i} + 1 - \beta \\
\frac{C_t}{A_t} &= \frac{\theta(p_t, \bar{c})}{A_t} + 1 - \beta
\end{aligned}
$$

This shows that $\theta(p_t, \bar{c})(a_t^i - A_t) > 0$ if and only if $\frac{c_t^i}{a_t^i} < \frac{C_t}{A_t}$, which happens if and only if $\frac{a_{t+1}^i}{a_t} > \frac{A_{t+1}}{A_t}$. That is, if $\theta > 0$ and an individual is wealthier than average, then their share of wealth grows faster than the growth of aggregate wealth. Equivalently, rich people save more. If $\bar{c} = 0$, then $\theta(p^t, \bar{c}) = 0$ and the wealth distribution is constant. In this case, the initial wealth distribution perpetuates itself.

Theorem 1.3 (Chatterjee, 1994.) *$\theta(p^t, \bar{c}) > 0$ if and only if $\bar{c}(K_t - K^*) > 0$. In a growing economy, $K_t < K^*$, and since $\bar{c} < 0$, we have $\theta(p^t, \bar{c}) > 0$. That is, the wealth distribution becomes more unequal as the economy grows.*

Proof *(Sketch.)* In a growing economy, $K_{t+1} > K_t$. Then, $\frac{p_t}{p_{t+1}} = f'(K_{t+1}) + 1 - \delta$. Thus, $\frac{p_{t+1}}{p_t}$ increases over time as $f'(K_t)$ declines. The limit of $\frac{p_{t+1}}{p_t}$ is β. Thus, $\frac{p_{t+1}}{p_t} < \beta$ for at some t along the transition. Then, remembering that $\bar{c} < 0$,

$$
\begin{aligned}
\theta(p^t, \bar{c}) &= \bar{c} \left((1-\beta) \sum_{\tau=t}^{\infty} \frac{p_\tau}{p_t} - 1 \right) \\
&> \bar{c} \left((1-\beta) \sum_{\tau=t} \beta^{\tau - t} - 1 \right) = 0
\end{aligned}
$$

■

All of these results rely on certain assumptions:

- *All agents have the same β.* If this fails, not only is there no aggregation result, but the more patient agents will save more each period until the agents with the highest β^i hold all the wealth in the steady state. (This is a degenerate wealth distribution.)

- *Complete markets.* On the other end of the spectrum, if there were autarky, each agent would maximize using his own production function. Since the agents are identical except for their initial $k_0^i > 0$, they will all converge to the same k^* which leads to perfect equality in the steady state. If instead they can trade, each person has the same return on capital instead. If people own capital individually and cannot trade it, all people will converge to owning the same amount of capital, but the initially poor must borrow from the initially wealthy to do this.

- We may add labor and have the same aggregation results if preferences are of the form $u(c_{it}, h_{it}) = \frac{1}{1-\sigma}(c_{it}^{\alpha}(1-h_{it})^{1-\alpha})^{1-\sigma}$.

Definition The steady state is *indeterminate* if observing the aggregates is not sufficient to determine the distribution. Equivalently, there are many possible steady states for the distribution, depending on the initial distribution.

It is also possible for there to be indeterminacy of the transition path, even if the eventual steady state is unique; that is not true in this particular model.

Theorem 1.4 *The wealth distribution in the steady state is indeterminate. That is, the wealth distribution depends on the initial conditions.*

Proof In the steady state, $\theta = 0$, so the wealth distribution is characterized by:

$$\begin{aligned}
c^i &= (1-\beta)a^i, \; i = 1, ..., N \\
a^i &= s^i \frac{1}{1-\beta}(f(K^*) - \delta K^*), \; 1 = 1, ..., N \\
A^* &= \sum_{\tau=t}^{\infty} \beta^{\tau-t}(f(K^*) - \delta K^*) = \frac{1}{1-\beta}(f(K^*) - \delta K^*) \\
f_K(K^*) + 1 - \delta &= \frac{1}{\beta} \\
\sum_{i=1}^{N} \mu_i s^i &= 1
\end{aligned}$$

This yields $2N+2$ equations for $3N+1$ variables, so that there is indeterminacy of dimension $N-1$. Since K^* (and therefore A^*) is pinned down by f_K, it must be (c^i, a^i, s^i) that are indeterminate. That is, there is a continuum of steady states of dimension $N-1$ that is consistent with K^*. ∎

Given the initial conditions, $a_0^1, ..., a_0^N$, the equilibrium wealth distribution is uniquely determined for all t, including the steady state.

Suppose productivity shifts from $z_L f(K)$ to $z_H f(K)$, with $z_L < z_H$. Then, K^*, A^* increase, which shifts the simplex of possible wealth distributions outward (but parallel). Since the economy grows after this shift, the resulting steady state wealth distribution will be less equal.

1.3 Negishi Approach

Suppose there are heterogeneous agents in a complete market. We may then use a modified planner problem to compute prices and quantities in a competitive equilibrium without requiring an aggregation result:

- each agent receives a time-constant weight in the utility function, and
- the weights are adjusted so that the initial endowments work out with no transfers needed to balance their budgets.

This uses the first welfare theorem which implies that every competitive equilibrium solves the social planner problem for some weights. This provides a mapping from the weights in the planner problem to initial endowments in competitive equilibrium. The Pareto weights now vary because the individual endowments differ.

If α^1 and α^2 are the weights on the agents, then the ratio of marginal utilities is constant and satisfies $\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\alpha^2}{\alpha^1}$. The weights in the planner problem are the inverses of the multipliers on the individuals' budget constraints; if the multiplier is small, this means that the shadow price of that agent's wealth is small, so that that agent has more wealth and therefore must have a high weight. For CRRA preferences, α^i will be a power of s_0^i.

If there is no closed form, we use the Negishi algorithm.

Algorithm: Negishi algorithm. For an economy with N agents and initial endowments $a_0^1, ..., a_0^N$:

1. Guess a vector of weights, $\overline{\alpha} = (\alpha^1, ..., \alpha^N)$.

2. Compute the corresponding allocations by solving, for each t:

$$\begin{aligned} \alpha^i \beta^t u'(c_t^i) &= \theta_t, \ i = 1, ..., N \\ \sum_{i=1}^{N} c_t^i + K_{t+1} &= f(K_t) + (1-\delta)K_t \\ \frac{\theta_t}{\theta_{t+1}} &= f_K(K_{t+1}) + 1 - \delta \end{aligned}$$

   where K_t and θ_t are given each period since K_0 is known and θ_0 can be normalized to 1.

3. Define a transfer function, $\tau^i(\overline{\alpha})$, $i = 1, ..., N$, by

$$\tau^i(\overline{\alpha}) = \sum_{t=0}^{\infty} \theta_t c_t^i(\overline{\alpha}) - \theta_0 a_0^i$$

   Since θ_t is proportional to p_t, this checks whether each agent's budget is balanced. If $\tau^i(\overline{\alpha}) > 0$, then α^i must be reduced. If $\tau^i(\overline{\alpha}) < 0$, then α^i must be increased.

4. Use the newly adjusted weights in step 2. Continue until all agents' budgets are balanced.

2 Income Fluctuation Problem

In the income fluctuation problem, an individual agent faces uncertainty in income and wants to allocate consumption. Throughout this section, we assume that $R = 1 + r$ is constant and exogenous, because this agent is too small to affect the economy and the economy is an a steady state overall.

Let $s_t \in S^t$ be the state of the economy at time t and $s^t = (s_0, ..., s_t) \in S^t = S_0 \times ... \times S_t$ be the history up to time t. Let $\pi(s^t)$ be the probability of history s^t and $y_t^i(s^t)$ be the endowment or income of the individual i upon the realization of history s^t. Then, the aggregate endowment is $Y_t(s^t) = \sum_{i=1}^{I} y_t^i(s^t)$. If this is time-varying, then there is *aggregate uncertainty.*

In autarky, $c_t^i(s^t) = y_t^i(s^t)$ for all i, t, s^t.

If markets are complete, then we have the time 0 budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t)(c_t^i(s^t) - y_t^i(s^t)) = 0$$

Solving the planner problem by maximizing $\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) \sum_{i=1}^{I} \alpha^i u(c_t^i(s^t))$ subject to $\sum_{i=1}^{I} c_t^i(s^t) = y_t(s^t)$ yields $c_t^i(s^t)$, according to the first order conditions

$$\alpha^i \beta^t \pi_t(s^t) u'(c_t^i(s^t)) = \theta_t(s^t)$$

For any agents, i, j, the marginal utilities satisfy $\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\alpha^j}{\alpha^i}$; this shows that there is *full risk sharing* or *complete insurance* across all times and histories.

For complete markets with log utility, the ratio of consumption will be constant, so that $c_t^j(s^t) = \frac{\alpha_j}{\sum_{i=1}^{N} \alpha_i} C_t(s^t)$. Thus, $\Delta c_t^j = \Delta C_t$, and the percentage change in individual consumption depends only on the change in aggregate consumption. In autarky, the percentage change in individual consumption depends only on the change in individual income. These two facts suggest an empirical test (assuming log utility) in which we estimate the model:

$$\Delta c_t^j = \beta_1 \Delta C_t + \beta_2 \Delta y_t^j + \epsilon_t^j$$

Under autarky, $\beta_1 = 0, \beta_2 = 1$, while under full insurance, $\beta_1 = 1, \beta_2 = 0$. Empirically, autarky is strongly rejected and full insurance is rejected, but less strongly. Thus, the truth probably lies somewhere in between.

In the income fluctuation problem, the asset space is limited to a single risk-free asset. Then, the budget constraint becomes:

$$c_t^i(s^t) + q_t(s^t) a_{t+1}^i(s^t) = y_t^i(s^t) + a_t^i(s^{t-1})$$

where $q_t(s^t)$ is the price of the single asset which pays one unit of consumption in the next period in any state. Because there are not complete markets, individuals cannot directly insure against bad income shocks; instead they hold extra assets as *self-insurance* or *precautionary savings*. For now, we assume that there are no aggregate fluctuations, which means that $q_t(s^t) = q = \frac{1}{1+r}$ for all histories. This leads to the simplified budget constraint:

$$a_{t+1} = (1+r)(y_t + a_t - c_t)$$

We also impose the No-Ponzi Scheme condition that $E_0\left(\lim_{t\to\infty}(\frac{1}{1+r})^t a_t\right) \geq 0$.

2.1 Permanent Income Hypothesis

Theorem 2.1 Permanent Income Hypothesis. *If preferences are quadratic and* $\beta(1+r) = 1$*, then consumptions follows a martingale process and equals the annuity value of human and financial wealth.*

Proof Suppose $u(c_t) = b_1 c_t - \frac{b_2}{2} c_t^2$ and $\beta(1+r) = 1$. Then, by the Euler equation:

$$\begin{aligned} b_1 - b_2 c_t &= \beta(1+r)E_t(b_1 - b_2 c_{t+1}) \\ c_t &= E_t(c_{t+1}) \end{aligned}$$

If we then iterate on the budget constraint, we find that:

$$\begin{aligned} \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j &= a_t + \sum_{j=0}^{\infty} E_t(y_{t+j}) = a_t + H_t \\ c_t &= \frac{r}{1+r}(a_t + H_t) \end{aligned}$$

which is the annuity value of the sum of current financial wealth and human wealth (which is the expected discount value of future income). ∎

Definition A policy function satisfies *certainty equivalence* if the solution to the stochastic problem is the same as a the solution of the non-stochastic problem with expectations of the exogenous variables substituted for the future values.

In the non-stochastic case,

$$c_t = \frac{r}{1+r}\left(a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right)$$

This shows that certainty equivalence holds in this case; this happens because of the linear-quadratic utility.

Theorem 2.2 *Under the permanent income hypothesis, consumption growth from periods t to $t+1$ is proportional to the change in expected earnings due to new information.*

Proof Let $W_t = a_t + H_t$. Then,

$$\begin{aligned} W_{t+1} - E_t(W_{t+1}) &= a_{t+1} - E_t(a_{t+1}) + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (E_{t+1}(y_{t+j}) - E_t(y_{t+j+1})) \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (E_{t+1}(y_{t+j}) - E_t(y_{t+j+1})) \end{aligned}$$

since there is no uncertainty today about next period's assets. Then,

$$\begin{aligned} \Delta c_{t+1} &= c_{t+1} - c_t = c_{t+1} - E_t(c_{t+1}) \\ &= \frac{r}{1+r}(W_{t+1} - E_t(W_{t+1})) \\ &= \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (E_{t+1}(y_{t+j}) - E_t(y_{t+j+1})) \end{aligned}$$

$\blacksquare$

Suppose that y_t depends on a martingale permanent component and an independent and identically distributed temporary component, so that:

$$\begin{aligned} y_t &= y_t^p + u_t \\ y_t^p &= y_{t-1}^p + v_t \end{aligned}$$

Then, $y_t = y_{t-1} + u_t - u_{t-1} + v_t$, and we may compute the change in expectations of future income for all future periods:

$$\begin{aligned} E_t(y_t) - E_{t-1}(y_t) &= y_t - (y_{t-1} - u_{t-1}) \\ &= (y_{t-1} + u_t - u_{t-1} + v_t) - (y_{t-1} - u_{t-1}) \\ &= u_t + v_t \\ E_t(y_{t+j}) - E_{t-1}(y_{t+j}) &= v_t \end{aligned}$$

Then, the change in consumption is:

$$\begin{aligned} \Delta c_t &= \frac{r}{1+r}\left(u_t + v_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j\right) \\ &= \frac{r}{1+r} u_t + v_t \end{aligned}$$

Thus, households respond weakly to transitory shocks and strongly to permanent shocks, since the latter shocks have a larger effect on permanent income. Though this model is very specific, we assume that, in general, consumption responds more to permanent income shocks than the transitory income shocks.

If this model held empirically, then we could compute:

$$\begin{aligned} Var_t(\Delta c_t) &= \left(\frac{r}{1+r}\right)^2 Var(u_t) + Var(v_t) \approx Var(v_t) \\ Var_t(\Delta y_t) &= Var(v_t) + 2Var(u_t) \end{aligned}$$

which would allow us to estimate the sizes of the two components.

2.2 Prudence in Finite Time Horizons

Definition The convexity of marginal utility, that is, the fact that $u''' > 0$, is called *prudence*. The *index of relative prudence* is given by $-\frac{u'''(c)c}{u''(c)}$.

If a utility function has decreasing absolute risk aversion (DARA), then it has a positive third derivative and will show prudence.

Theorem 2.3 *If the individual is "prudent", then a rise in future income uncertainty leads to an increase in current saving and a decline in current consumption.*

Proof Consider the two-period consumption problem of maximizing $u(c_0) + \beta E(u(c_1))$ according to the budget constraints:

$$\begin{aligned} c_0 + a_1 &= a_0 + y_0 \\ c_1 &= Ra_1 + y_1 \end{aligned}$$

where a_0, y_0 are known, but y_1 is stochastic and exogenous. If we assume that $\beta R = 1$, then the Euler equation becomes:

$$u'(a_0 + y_0 - a_1) = E\left(u'(Ra_1 + y_1)\right)$$

As long as utility is concave, the left-hand-side is increasing in a_1 while the right-hand-side is decreasing in a_1, so that there is a unique solution for a_1. Higher saving implies lower period-zero consumption.

Consider a mean-preserving spread of y_1, $\tilde{y}_1 = y_1 + \epsilon_1$, with $E(\epsilon_1) = 0$, $Var(\epsilon_1) = \sigma_\epsilon^2$, and ϵ_1 independent of y_1. Then, the Euler equation becomes:

$$u'(a_0 + y_0 - a_1) = E\left(u'(Ra_1 + y_1 + \epsilon_1)\right)$$

If u' is convex, then we may apply Jensen's inequality (after using the law of iterated expectations to condition on y_1) to show that adding ϵ_1 will increase the value of the right-hand-side, so that the optimal a_1 will increase while the optimal c_0 will fall. ∎

The additional savings induced by prudence are called *precautionary savings* or *self-insurance*. The amount of precautionary savings can be measured by comparing asset holdings under no uncertainty to the computed asset holdings.

This method may be extended to the multiple period case using the following dynamic programming formulation:

$$\begin{aligned} V^t(a,y) &= \max_{c_t,a_{t+1}} \left(u(c_t) + \beta E\left(V^{t+1}(a_{t+1}, y_{t+1})\right)\right) \\ c_t + a_{t+1} &= Ra_t + y_t \end{aligned}$$

If y_t is independent and identically distributed (so that knowing y gives no information about y'), then we may reduce the state space by using only the state variable *cash-in-hand*, $x_t = Ra_t + y_t$, which leads to the formulation:

$$\begin{aligned} V^t(x) &= \max_{c_t,x_{t+1}} \left(u(c_t) + \beta E\left(V^{t+1}(x_{t+1})\right)\right) \\ x_t &= c_t + a_{t+1} \\ x_{t+1} &= R(x_t - c_t) + y_{t+1} \end{aligned}$$

Using cash-in-hand, the Euler equation is:

$$u_1(c_t) = E\left(V_1^{t+1}(R(x_t - c_t) + y_{t+1})\right)$$

Then, precautionary savings will occur if $\frac{d^3}{dx^3}V^{t+1} > 0$. If T is finite, then one can show that $u''' > 0$ implies that $\frac{d^3}{dx^3}V^{t+1} > 0$.

2.3 Borrowing Constraints

Definition Let ϕ be any non-negative fixed number. The requirement that $a_{t+1} \geq -\phi$ is an *ad-hoc borrowing constraint*.

There will be precautionary saving with quadratic utility and borrowing constraints. In this case, assets can be computed using:

$$\Delta a_t = -\sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^{j-1} E_t(\Delta y_{t+j})$$

If $\Delta y_{t+j} = \epsilon_{t+j}$, so that income is a random walk, then $\Delta a_t = 0$, and agents will have constant financial wealth (always consuming only the interest). However, if $y_{t+j} = \epsilon_{t+j}$, so that income is independent and identically distributed, then $\Delta y_{t+j} = \epsilon_{t+j} - \epsilon_{t+j-1}$ and $\Delta a_t = \epsilon_t$ and wealth is a random walk. Then, there is a positive probability of hitting any exogenous borrowing constraint. This means that the agent cannot always act optimally, and will save in anticipation of this.

Suppose the borrowing constraint is $a_{t+1} \geq 0$, and we use the budget constraint $a_{t+1} = R(a_t + y_t - c_t)$. When the constraint binds, we know that $a_{t+1} = 0$ and $c_t = a_t + y_t$, so that the Euler equation must be:

$$\begin{aligned} c_t &= \begin{cases} E_t(c_{t+1}) & a_{t+1} > 0 \\ y_t + a_t & a_{t+1} = 0 \end{cases} \\ c_t &= \min\{y_t + a_t, E_t(c_{t+1})\} \\ &= \min\{y_t + a_t, E_t(\min\{y_{t+1} + a_{t+1}, E_{t+1}(c_{t+2})\})\} \end{aligned}$$

If there is a mean-preserving spread of y_{t+1}, then c_t will decrease in some cases, because $c_{t+1} = y_{t+1} + a_{t+1}$ is more likely to be binding, which will lead to lower consumption with a positive probability. Thus, the agent will consume less and save more in the present to self-insure against bad income shocks.

Definition Suppose $\{y_t\}_{t=0}^{\infty}$ is deterministic. Then, the *natural borrowing constraint* sets consumption to 0 in all future periods:

$$\begin{aligned} a_t &\geq -y_t + \frac{a_{t+1}}{R} \\ &\geq \ldots \\ &\geq -\sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j y_{t+j} \end{aligned}$$

That is, the individual cannot borrow more than can be paid back, using the discounted income stream. If $\{y_t\}_{t=0}^{\infty}$ is stochastic, then the borrowing constraint ensures that the individual can repay the debt in the worst state of the world. Let y_{min} be the minimum realization of income (which we assume is constant through time). Then, the borrowing constraint is:

$$a_t \geq -\sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j y_{min} = -\frac{1+r}{r} y_{min}$$

(the timing of interest may change $\frac{1+r}{r}$ to $\frac{1}{r}$). If the agent also chooses a labor supply, the natural borrowing constraint also assumes that they work as much as possible to maximize their income.

As long as $u(0) = -\infty$, the natural borrowing constraint is never binding, since the agent would never allow $c_t = 0$ to occur with positive probability. Many bad realizations of y_t will push the agent closer to the natural borrowing constraint.

2.4 The General Infinite Horizon Income Fluctuation Problem

For more general utility functions, let $\{c_t^*\}$ be the optimal consumption sequence, with the no-borrowing constraint, $a_t \geq 0$. We assume that y_t doesn't go to ∞. Then, if c_t^* diverges,

a_t must diverge as well, in order to finance the consumption. If there were a continuum of agents in the economy making the same decision, this result could not be an equilibrium, and r would not be an equilibrium interest rate. Conversely, if c_t^* is bounded above, then $a_{t+1} \in [0, \overline{a}]$, and the domain of asset holdings is bounded above.

The general income fluctuation problem is to maximize $E_0(\sum_{t=0}^{\infty} \beta^t u(c_t))$ subject to $c_t + a_{t+1} = (1+r)a_t + y_t$ (the timing of interest changes here a little) and $a_{t+1} \geq 0$. We assume that $u' > 0, u'' < 0$ and that the Inada conditions hold for the utility function. Then, the Euler equation at each period is:

$$u'(c_t) = \beta(1+r)E_t(u'(c_{t+1})) + \lambda_t$$

where $\lambda_t \geq 0$ is the multiplier on the borrowing constraint. Equivalently, we could write $u'(c_t) \geq \beta(1+r)E_t(u'(c_{t+1}))$, with equality when the borrowing constraint is not binding.

We consider three cases for $\beta(1+r)$ and $\{y_t\}$ non-stochastic:

- Case $\beta(1+r) > 1$: Then, $u'(c_t) > u'(c_{t+1})$, and we must have $c_{t+1} > c_t$ by decreasing marginal utility. Then, $a_{t+1} > a_t$ for all t; agents are saving because the interest rate (and therefore the return on saving) is high, relative to their impatience (as measured by β).

- Case $\beta(1+r) = 1$: Then, $u'(c_t) \geq u'(c_{t+1})$, and we must have $c_{t+1} \geq c_t$ so that consumption is a non-decreasing sequence, which is increasing if and only if the borrowing constraint is binding. Then, there must be some τ such that c_t is flat for all $t > \tau$, and the budget constraint will be active only up to time τ. Let $H_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j}$, which is human wealth. Then, τ is the date when x_t is maximized; this occurs because people want to borrow and spend in anticipation of future wealth, which causes the borrowing constraint to be binding.

- Case $\beta(1+r) < 1$: In this case, the consumption sequence will converge.

As a special case, suppose $y_t = y$ is constant. It does not necessarily follow that consumption is constant. Let $x = (1+r)a + y$. Then, using dynamic programming:

$$\begin{aligned} V(x) &= \max_{c,a'} (u(c) + \beta V(x')) \\ c + a' &= x \\ x' &= (1+r)(x-c) + y \\ a' &\geq 0 \end{aligned}$$

By the envelope theorem, $u_c(c) = V_x(x)$. Assuming that the value function is twice-differentiable and concave:

$$\begin{aligned} u_{cc}(c)\frac{dc}{dx} &= V_{xx}(x) \\ \frac{dc^*}{dx} &= \frac{V_{xx}(x)}{u_{cc}(c)} \end{aligned}$$

This means that $\frac{dc^*}{dx} > 0$ (since both second derivatives are negative), and consumption is increasing in cash-in-hand. Using the first order conditions and the envelope theorem, we find that:

$$\begin{aligned}
u_c(c) &= \beta(1+r)V_x(x') \\
V_x(x) &= \beta(1+r)V_x(x') < V_x(x')
\end{aligned}$$

Then, $x' < x$ by the strict concavity of V. Therefore, then $a > 0$, cash-in-hand will fall over time and consumption must fall, too, since consumption is increasing in cash-in-hand. Suppose that $x = y$ (that is, $a = 0$) but $a'(x) > 0$. Then, $a'(x) = a'(y) > 0$ and the Euler equation holds with equality since the borrowing constraint is not binding. Then:

$$\begin{aligned}
u'(c) &= \beta(1+r)V_x(x') \\
V_x(y) &= \beta(1+r)V_x\left((1+r)a'(y)+y\right) \\
&< V_x\left((1+r)a'(y)+y\right) \\
&< V_x(y)
\end{aligned}$$

(since $a'(y) > 0, \beta(1+r) < 1$) which is a contradiction. Thus, when $x = y$, $a'(y) = 0$ and $c(y) = y$. Combining the two results shows that if $a > 0$, consumption will start high and then decline until it is constant at $c = y$. This completely describes the consumption sequence. This shows that if $a_0 > 0$, assets are decumulated until $a = 0$; then the agent consumes only income.

Definition A *supermartingale* is a random variable, $\{M_t\}$, that satisfies $M_t \geq E_t(M_{t+1})$.

Theorem 2.4 Supermartingale Convergence Theorem (Doob). *A non-negative supermartinagle converges almost surely to a non-negative random variable with bounded support, $\overline{M}$, with $E(|\overline{M}|) < \infty$.*

In the stochastic case, income uncertainty may lead to additional savings for precautionary reasons. This will lead to more stringent requirements for r. Let $M_t = (\beta(1+r))^t u'(c_t)$. This is a non-negative supermartingale, since $M_t \geq E_t(M_{t+1})$ by the Euler equation.

- Case $\beta(1+r) > 1$: By the supermartingale theorem, $(\beta(1+r))^t u'(c_t)$ has a finite limit. Since $(\beta(1+r))^t \to \infty$, $u'(c_t) \to 0$, so that $c_t \to \infty$ (and $a_t \to \infty$).
- Case $\beta(1+r) = 1$: Assume that $u'''(c) > 0$. Then, by Jensen's inequality, $u'(c_t) = E_t(u'(c_{t+1})) > u'(E_t(c_{t+1}))$ and $E_t(c_{t+1}) > c_t$. Thus, expected consumption diverges in this case. In general, it can be shown that consumption diverges as well.
- Case $\beta(1+r) < 1$: There are not general conditions for the asset space to be bounded. One that is sufficient is for the shocks to be IID when the agent has decreasing absolute risk aversion.

Proposition 2.5 *For IID shocks, $\beta(1+r) < 1$ is a necessary but not sufficient condition to ensure that consumption converges. If $\beta(1+r) < 1$ and there is decreasing absolute risk aversion (equivalently, prudence), then consumption converges.*

Proof Let x be cash-in-hand. By the Euler equation:

$$\begin{aligned} u_c(c(x)) &= \beta R E(u_c(c(x'))) \\ &= \beta R \frac{E(u_c(c(x')))}{u_c(c_{max}(x))} u_c(c_{max}(x)) \end{aligned}$$

where $c_{max}(x) = c(Ra'(x)+y_{max})$ is the consumption choice associated with the best shock next period. Suppose that $\lim_{x\to\infty} \frac{E(u_c(c'(x')))}{u_c(c_{max}(x))} \leq 1$. Then, for large enough x,

$$u'_c(c(x)) = \beta R u'_c(c_{max}) < u'_c(c_{max}(x))$$

so that $c_{max}(x) < c(x)$ implies that $x_{max} < x$ by the concavity of u. This leads to an upper bound on the asset space, $\overline{x}$, since cash-in-hand does not increase forever. We assume that $\overline{x}$ exists (?) such that

$$\frac{E(u_c(c'(\overline{x})))}{u_c(c_{max}(\overline{x}))} \leq 1$$

Thus implies an x'_{max}. Using a Taylor series approximation of $u_c(c(x'))$ about x'_{max}, taking expectations, and manipulating, we find that:

$$\begin{aligned} u_c(c'(x')) &\approx u_c(c(x'_{max})) + u_{cc}(c(x'_{max}))c(x'_{max})(x' - x'_{max}) \\ E(u_c(c'(x'))) &\approx u_c(c(x'_{max})) + u_{cc}(c(x'_{max}))c(x'_{max})E(x' - x'_{max}) \\ \frac{E(u_c(c'(x')))}{u_c(c(x'_{max}))} &\approx 1 - \frac{u_{cc}(c(x'_{max}))}{u_c(c(x'_{max}))} c(x'_{max})E(x' - x'_{max}) \end{aligned}$$

Note that $E(x' - x'_{max})$ is positive and finite because it is simply $E(y' - y'_{max})$ and the distribution of y has a finite mean. Also, $c(x'_{max})$ is positive and finite. Taking the limit as $x \to \infty$, we find that the limit will be 1 if $\frac{u_{cc}(c(x'_{max}))}{u_c(c(x'_{max}))} \to 0$; this term is the coefficient of absolute risk aversion. In this case, consumption will converge. ∎

With DARA, the individual becomes less risk averse with more wealth accumulation and therefore needs less precautionary saving. The faster this coefficient falls, the lower the bound on the asset space (which is good for numerical solution methods).

There are other kinds of savings in reality, such as savings for life-cycle reasons.

2.5 Numerical Solution Methods

2.5.1 State Space Discretization

Suppose we have a Markov process for income in which the logarithm is an AR(1) process:

$$\begin{aligned} y_t &= \log(Y_t) \\ y_t &= \rho_t y_{t-1} + \epsilon_t \\ \epsilon_t &\sim G(0, \sigma_\epsilon^2) \end{aligned}$$

Instead of using a continuous state space for income (which makes computation more difficult), we discretize the state space and use a Markov chain for income with a finite number of values.

Let F be the standardized version of G, so that $G(\epsilon) = F(\frac{\epsilon}{\sigma_\epsilon})$.

We first choose N values, $\overline{y}_1, ..., \overline{y}_N$, to make up the Markov chain. Larger values of N lead to better approximations, but even $N = 9$ can be good enough. To choose the values once N has been chosen:

- Choose $\overline{y}_N = m\sqrt{\frac{\sigma_\epsilon^2}{1-\rho^2}}$, with m large enough to capture most of the total mass of the distribution; $m = 3$ is reasonable for a normal distirbution.
- Set $\overline{y}_1 = -\overline{y}_N$ (assuming that the distribution is symmetric).
- Set $\overline{y}_2, ..., \overline{y}_{N-1}$ to be equidistant between the two endpoints. (One could also use *Gaussian quadrature* or the *collocation method* to choose the point spacing.)

Set d to be the distance between points. We then compute the transition probabilities for interior k as:

$$\begin{aligned} \pi_{jk} &= P\left(\overline{y}_k - \frac{d}{2} \leq \rho\overline{y}_j + \epsilon_{t+1} \leq \overline{y}_k + \frac{d}{2}\right) \\ &= F\left(\frac{\overline{y}_k + \frac{d}{2} - \rho\overline{y}_j}{\sigma_\epsilon}\right) - F\left(\frac{\overline{y}_k - \frac{d}{2} - \rho\overline{y}_j}{\sigma_\epsilon}\right) \end{aligned}$$

At the endpoints, we have the transition probabilities:

$$\begin{aligned} \pi_{j1} &= F\left(\frac{\overline{y}_1 + \frac{d}{2} - \rho\overline{y}_j}{\sigma_\epsilon}\right) \\ \pi_{jN} &= 1 - F\left(\frac{\overline{y}_N - \frac{d}{2} - \rho\overline{y}_j}{\sigma_\epsilon}\right) \end{aligned}$$

As $N \to \infty$ and $d \to 0$, this converges to the original continuous process.

To assess the accuracy of this approximation:

- Simulate from the process and compute moments such as the unconditional variance, the covariances. Compare them to the moments of the continuous process. If the moments do not match closely enough, increase N or adjust the spacing of the points.

- Let P be the transition matrix. Then, we may compute the stationary distribution, π^*, as the vector that solves $\pi^* = P\pi^*$. From the invariant distribution and the transition matrix, one can compute moments exactly and compare them to the moments of the original process.

2.5.2 Policy Function Iteration

Since the income fluctuation problem must have a bounded solution for certain interest rates, numerical solution methods are feasible.

To solve the problem, we must find policy functions, c, a', in terms of the state variables of assets and income. In general, we discretize the income process using the method above, so that there is only one continuous variable.

The discretized recursive formulation of the agent's problem is:

$$V(a, y) = \max_{c,a':c+a'=Ra+y,a'\geq-\phi} \left(u(c) + \beta \sum_{i=1}^{N} \pi(y_i, y) V(a', y_i) \right)$$

This solution is in the form $a'(a, y)$, which implies $c(a, y)$.

In general, there are three options for function approximation:

- *Local Approximation*: This approximates the function in the neighborhood of a point. This works well for a stochastic growth model where there is an obvious point to approximate about (the steady state), where deviations from that point tend to be relatively small, and where u''' doesn't matter. This is done with log-linearization or a linear-quadratic approximation. It is generally not appropriate for this problem, since there is no obvious base point, income tends to vary more, and u''' matters.

- *Global Approximation*: This approximates a function over a range. One way is through pure discretization, in which the asset space is discretized into $a_1, ..., a_m$ and the policy function is undefined anywhere that is not those points. Alternatively, one can use local linear interpolation in which the policy function is approximated by a line (or higher order polynomial) between grid points. Both approximations work better if there are more points where the function is most curved (in this case, near the borrowing constraint). In either case, a larger M is more accurate but more time-consuming.

- *Global Interpolation*: One can approximate the entire function with a single high-degree polynomial (such as a Chebyshev polynomial). This method will lead to a smooth policy function, but the function might not be concave.

The approximation can be applied to either the value function or the policy function; we use the policy function.

Recall that the Euler equation is:

$$u'(Ra + y - a') - \beta R \sum_{i=1}^{N} \pi(y_i, y) u'(Ra' + y' - a'') \geq 0$$

where equality holds if $a' > -\phi$. In this algorithm, we guess $a''(a, y)$, solve for the $a'(a, y)$ implied by this guess, then set $a'' = a'$ and iterate until the function converges.

Algorithm: Pure Discretization Policy Function Approximation

1. Construct a grid on the asset space, $a_1, ..., a_M$, with $a_1 = -\phi$, a_M large enough that it is likely to contain all the possible asset values, and the intermediate points spaced so that they are closer together near a_1.

2. Guess a policy function $a''(a, y) = \hat{a}_0(a_i, y_j)$, $i = 1, ..., M; j = 1, ..., N$, where $\hat{a}_0(a_i, y_j)$ is always on the grid. (For example, one could guess that $\hat{a}_0(a_i, y_j) = a_i$.)

3. At each (a_i, y_j), check if

$$u'(Ra_i + y_j - a_1) - \beta R \sum_{y' \in Y} \pi(y', y_j) u' \left(Ra_1 + y' - \hat{a}_0(a_1, y') \right) > 0$$

   If the above equation is true, then the budget constraint is binding at this point, and we set $\hat{a}_1(a_i, y_j) = a_1$.

4. Otherwise, we know that the budget constraint is not binding, and we search for a consecutive pair, (a_k, a_{k+1}), such that:

$$\delta(a_k) = u'(Ra_i + y_j - a_k) - \beta R \sum_{y' \in Y} \pi(y', y') u \left(Ra_k + y' - \hat{a}_0(a_k, y') \right) < 0$$

   and $\delta(a_{k+1}) > 0$. By the monotonicity of this problem, the pair will be unique. Set $\hat{a}_1(a_i, y_j) = \arg\min\{|\delta(a_k)|, \delta(a_{k+1})\}$ (this is the asset choice that is closer to exactly satisfying the Euler equation).

5. Check for convergence by checking whether $\hat{a}_1(a_i, j_j) = \hat{a}_0(a_i, y_j)$ for all i, j (since the problem is discrete, equality is a reasonable thing to require). If the two are not equal, set $\hat{a}_0 = \hat{a}_1$ and return to step 3.

Algorithm: Piecewise Linear Policy Function Approximation

1. Construct a grid on the asset space, $a_1, ..., a_M$, with $a_1 = -\phi$, a_M large enough that it is likely to contain all the possible asset values, and the intermediate points spaced so that they are closer together near a_1.

2. Guess a policy function $a''(a,y) = \hat{a}_0(a_i, y_j)$, $i = 1, ..., M; j = 1, ..., N$, where $\hat{a}_0(a_i, y_j) \in [-\phi, a_M]$ can be any value (not necessarily on the grid).

3. At each (a_i, y_j), check if

$$u'(Ra_i + y_j - a_1) - \beta R \sum_{y' \in Y} \pi(y', y_j) u'\left(Ra_1 + y' - \hat{a}_0(a_1, y')\right) > 0$$

If the above equation is true, then the budget constraint is binding at this point, and we set $\hat{a}_1(a_i, y_j) = a_1$.

4. Otherwise, we know that the budget constraint is not binding, and we solve the non-linear Euler equation for a^*, the optimal savings today:

$$u'(Ra_i + y_j - a^*) - \beta R \sum_{y' \in Y} \pi(y', y) u'\left(Ra^* + y' - \hat{a}_0(a^*, y')\right) = 0$$

where we assume that $\hat{a}_0$ is piecewise linear in a between the gridpoints. That is, if $a_k < a < a_{k+1}$ then $\hat{a}_0(a,y) = \hat{a}_0(a_k, y) + (a - a_k)\frac{\hat{a}_0(a_{k+1}) - \hat{a}_0(a_k)}{a_{k+1} - a_k}$. Then, set $\hat{a}_1(a_i, y_j) = a^*$.

5. Check for convergence by checking whether $\max_{i,j} |\hat{a}_1(a_i, y_j) - \hat{a}_0(a_i, y_j)| < \epsilon$ for some small ϵ. If the two are not equal, set $\hat{a}_0 = \hat{a}_1$ and return to step 3.

If the solution is not accurate enough, one may need to add more points to the grid of assets or increase the order of the approximating polynomial.

One way to assess the accuracy (Denhaan and Marcet, 1994) is to note that:

$$\begin{aligned} u'(c_t) &= \beta R E_t(u'(c_{t+1})) \\ &= \beta R u'(c_{t+1}) + \epsilon_{t+1} \end{aligned}$$

where ϵ_t is the expectation error. Then, ϵ_{t+1} should be uncorrelated with any variable known at time t, such as y_s, c_s, a_s or any function of these variables for any $s \leq t$. We write these functions of predetermined variables as the $r \times 1$ vector, $h(z_t)$. Then, at the exact solution, $E_t(\epsilon_{t+1} \otimes h(z_t)) = 0$, where $\otimes$ is element-by-element multiplication. We may simulate to compute the expectation. Let $\hat{B}_S = \frac{1}{S}\sum_{s=1}^{S} \hat{\epsilon}_{s+1} \otimes h(\hat{z}_s)$, where S is the number of simulations, $\hat{\epsilon}_{s+1}$ is the error from the approximate Euler equation, and z_s are the realizations of the variables from the simulation. If there is truly independence, then $\sqrt{S}\hat{B}_S \rightarrow_D Normal(0, V)$, so that we may apply a ξ_r^2 test to $S\hat{B}_S\hat{V}_S^{-1}\hat{B}_S$ to check if B_S differs from 0. However, as $S \rightarrow \infty$, the approximation will be rejected almost surely (because it is never exact); choosing S to be a multiple of r is usually more realistic. Comparing $S\hat{B}_S\hat{V}_S^{-1}\hat{B}_S$ across different solution methods can also be helpful in assessing relative accuracy.

Alternatively, we may use the Euler equation errors directly. At the solution, $u'(c_t) = \beta R E_t(u'(c_{t+1}))$. For each t, we may express the error as a fraction of consumption:

$$
\begin{aligned}
u'\left(c_t(1-\epsilon_t)\right) &= \beta R E_t\left(u'(c_{t+1})\right) \\
\epsilon_t &= 1 - \frac{(u')^{-1}\left(\beta R E_t(u'(c_{t+1}))\right)}{c_t}
\end{aligned}
$$

Then, ϵ_t is the error that the agent would be making, in terms of percent of consumption, by using the computed policy function. We compute $E_t(u'(c_{t+1})) = \sum_{y' \in Y} u'(c(y'))\pi(y'|y_t)$. We may simulate for S periods and analyze the properties of ϵ_t. In general, the average $|\epsilon_t|$ should be less than 10^{-4}. Another statistic of interest may be $\max(\epsilon_t)$. Ideally, ϵ_t should be white noise.

3 Measure Theory

Definition Let S be a set and $\mathcal{S}$ be a family of subsets of S. We call $\mathcal{S}$ a *σ-algebra* if:

- $\emptyset, S \in \mathcal{S}$,
- if $A \in \mathcal{S}$ then $A^C \in \mathcal{S}$, and
- if $A_n \in \mathcal{S}$ for $n = 1, ..., N$, then $\bigcup_{n=1}^{N} A_n \in \mathcal{S}$.

Then, $(S, \mathcal{S})$ is called a *measurable space* and each $A \in \mathcal{S}$ is called a *measurable set*.

Definition Let $\mathcal{A}$ be the collection of all open intervals in R, that is, all intervals of the form $(a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$. The smallest σ-algebra that contains $\mathcal{A}$ is the *Borel σ-algebra*. In R^n, the Borel σ-algebra is generated by the open balls.

Definition Let $(S, \mathcal{S})$ be a measurable space. A *measure* is a function, $\lambda : \mathcal{S} \to R$, such that:

- $\lambda(\emptyset) = 0$,
- $\lambda(A) \geq 0$ for all $a \in \mathcal{S}$, and
- if $\{A_n\}_{n=1}^{\infty}$ are in $\mathcal{S}$ and are disjoint, then $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n)$.

We call $(S, \mathcal{S}, \lambda)$ a *measure space*. If $\lambda(S) = 1$, then we call λ a *probability measure*.

Definition We call $f : S \to R$ a *measurable function* if, for all $a \in R$, $\{s \in S : f(s) < a\} \in \mathcal{S}$.

Definition A *transition function* is a function, $Q : S \times \mathcal{S} \to [0, 1]$ that satisfies:

- for each $s \in S$, $Q(s, .)$ is a probability measure on $(S, \mathcal{S})$, and
- for each $A \in \mathcal{S}$, $Q(., A)$ is a measurable function.

We think of $Q(s^*, A)$ as a conditional probability function, with

$$Q(s^*, A) = P(s' \in A | s = s^*)$$

so that Q measures the probability of moving from s^* to an element of A.

In macro models, Q depends on the exogenous law of motion, π, and the endogenous policy function, a'. For example, we might have:

$$Q\left((a^*, y^*), (A, \hat{y})\right) = \pi(\hat{y}|y^*) 1\left(a'(a^*, y^* \in A)\right)$$

Definition Let f be measurable and non-negative. We define the associated Markov operator of f by:

$$Tf(s) = \int_S f(s') Q(s, ds')$$

Notice that $Tf(s) = E(f(s')|s)$.

Definition For any probability measure, λ, define:

$$T^*\lambda(A) = \int_S Q(s, A) d\lambda(s)$$

Then, $T^*\lambda$ is the probability measure one period ahead, given the transition function and the probability measure today.

Definition A transition function, Q, has the *Feller property* if the associated Markov operator, T maps the space of continuous and bounded functions, $C(S, \mathcal{S})$, to itself. That is, T preserves boundedness and continuity.

Definition Q on $(S, \mathcal{S})$ is *monotone* if whenever f is increasing, Tf is also increasing. That is, T preserves monotonicity.

Definition λ is an *invariant distribution* if $\lambda = T^*\lambda$. That is, $\lambda(A) = \int_S Q(s, A)\lambda(ds)$ for all $A \in \mathcal{S}$.

Theorem 3.1 *If S is compact and Q has the Feller property, then there is at least one invariant distribution.*

This theorem does not imply uniqueness. There may be multiple ergodic sets. Also, the theorem does not give any indication about the speed of convergence.

Definition Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of probability measures with λ_0 and $\lambda_{n+1} = T^*\lambda_n$. (That is, $\lambda_n(A) = \int_S Q(s, A)\lambda_{n-1}(ds)$, for a given and fixed Q.) We say that $\{\lambda_n\}_{n=0}^{\infty}$ *converges weakly* to λ if

$$\lim_{n\to\infty} \int f d\lambda_n(A) \to \int f d\lambda(A)$$

for all $f \in C(S, \mathcal{S})$.

Definition Let $S = [a, b]$ be a bounded interval. We say that Q satisfies the *monotone mixing condition* if there exists $\hat{s} \in S, \epsilon > 0, N > 0$ such that $Q^N(a, [\hat{s}, b]) \geq \epsilon$ and $Q^N(b, [a, \hat{s}]) \geq \epsilon$, where Q^N is the application of Q N times. That is, there is a positive probability of going from the minimum or maximum of any bounded interval to the other end of the interval.

In the context of income fluctuation, this is sometimes called the *American Dream-American Nightmare* condition.

Theorem 3.2 *Let $S = [a, b]$ be a bounded interval and $\mathcal{S}$ the associated Borel σ-algebra. If Q satisfies the Feller property, is monotone, and satisfies the monotone mixing condition, then:*

- *Q admits a unique invariant probability measure, λ, and*
- *$(T^*)^n\lambda_0$ weakly converges to λ for any λ_0.*

This theorem yields a way to compute the invariant distribution, λ. In practice, it is helpful to try multiple initial distributions for the computation, especially if some of the conditions are too hard to prove.

4 Equilibrium with a Continuum of Agents and Idiosyncratic Risk

This model combines the income fluctuation problem with an aggregate production function to compute an equilibrium in the asset market. Most of the model is based on Aiyagari (1994). The model is called the *Standard Incomplete Market (SIM)* model, the Aiyagari model, the Imrohoroghu-Huggett-Aiyagari model, or the Bewley model.

In this economy:

- **Demographics**: There is a continuum of agents of measure 1, infinitely lived and identical ex ante.
- **Preferences** are given by $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ with $0 < \beta < 1$, $u' > 0$, and $u'' < 0$. These preferences assume that labor supply is inelastic (this can be generalized).

- **Endowments**: Individuals start with no assets. Each period, they have a stochastic endowment of ϵ_{it} *efficiency units of labor*, according to the Markov chain on the set $E = \{\epsilon^1, ..., \epsilon^N\}$ with transition probabilities, $\pi(\epsilon'|\epsilon)$. Shocks are independent and identically distributed across individuals (but not across time), so that $\pi(\epsilon'|\epsilon)$ is the fraction of agents moving from ϵ to ϵ' each period. (If there were correlation across agents, this would lead to aggregate shocks and therefore aggregate uncertainty.) We assume that the Markov chain has a unique invariant distribution, $\pi^*(\epsilon)$, which gives the fraction of agents at each efficiency level in the stationary state. This means that the aggregate labor supply in terms of efficiency units is:

$$H_t = \sum_{j=1}^{N} \epsilon_j \pi^*(\epsilon_j) = H$$

  which is constant in the stationary state.

- **Budget Constraint**:

$$\begin{aligned} c_t + a_{t+1} &= (1 + r_t)a_t + w_t\epsilon_t \\ a_{t+1} &\geq -b \end{aligned}$$

  Note that agents have both capital and labor income, and there is a borrowing constraint.

- **Aggregate technology** is given by the constant return to scale production function $F(K_t, H_t)$. Capital depreciates at rate $0 < \delta < 1$.

- **Market structure**: All markets are competitive.
  - In the labor market, the wage will adjust so that firms will always demand the aggregate supply of labor, H.
  - In the goods market, we normalize the price of the good to 1.
  - There is an assets market in which households save and in which firms get assets to buy capital.

- The **aggregate resource constraint** is

$$F(K_t, H_t) = C_t + I_t = C_t + K_{t+1} - (1 - \delta)K_t$$

In this model, we assume that there is an upper limit, $\overline{a}$, on assets. Then, individuals are distributed over the state space, $S = [-b, \overline{a}] \times E$. We define B_S to be the Borel σ-algebra on S, so that (S, B_S) is a measurable space. Then, for any measurable set, $\mathcal{A} \times \mathcal{E}$, we define the transition function, Q, by:

$$Q((a, \epsilon), \mathcal{A} \times \mathcal{E}) = \sum_{\epsilon' \in \mathcal{E}} \pi(\epsilon'|\epsilon) 1 \left(a'(a, \epsilon) \in \mathcal{A}\right)$$

This model can be used to answer questions about wealth inequality due to income fluctuations, the proportion of the capital stock due to precautionary savings, the redistributive effects of policies (and the tradeoff between insurance and inefficiency), the equity premium puzzle, and the role of labor supply in self-insurance.

4.1 The Steady State

Definition A *stationary recursive competitive equilibrium* consists of:

- a value function, $v : S \to R$, and associated household policy functions, $a' : S \to R$ and $c : S \to R_+$,
- policies for the firm, H, K,
- prices, r, w, and
- a stationary measure, λ^*, on $A \times E$,

such that:

- v is the solution to the household problem, given r, w, λ:

$$\begin{aligned} v(a,\epsilon) &= \max_{c,a'} \left(u(c) + \beta \sum_{\epsilon' \in E} v(a',\epsilon')\pi(\epsilon'|\epsilon) \right) \\ c + a' &= Ra + w\epsilon \\ a' &\geq -b \end{aligned}$$

  and a', c are the associated policy functions,

- given r and w, K and H are the optimal choices of the capital and labor input for the firm, so that:

$$\begin{aligned} F_K(K,H) &= r + \delta \\ F_H(K,H) &= w \end{aligned}$$

- the labor market clears: $\sum_{\epsilon \in E} \pi^*(\epsilon) = F_H^{-1}(w; K) = H$,
- the goods market clears: $F(K,H) = \delta K + \int_{A \times E} c(a,\epsilon) d\lambda^*(a,\epsilon)$ (we have embedded the steady state result that investment always equals depreciation in the steady state),
- the asset market clears:

$$K(r) = F_K^{-1}(r + \delta; H) = \int_{A \times E} a'(a,\epsilon) d\lambda^*(a,\epsilon) = A(r)$$

- the distribution of assets and endowments is stationary, so that, for all $\mathcal{A} \times \mathcal{E} \in B_S$,

$$\lambda^*(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q((a,\epsilon), \mathcal{A} \times \mathcal{E}) d\lambda^*(a,\epsilon)$$

In this model, the distribution of individuals is constant, but each individual is moving around.

To check these conditions, we note that equilibrium in the labor market is trivial when households have an inelastic labor supply and that equilibrium in the assets market implies equilibrium in the goods market. Thus, it is sufficient to find an equilibrium in the asset market.

We may take a graphical approach. Consider $A(r) = \int_{A \times E} a'(a,\epsilon) d\lambda^*(a,\epsilon)$ and $K(r) = F_K^{-1}(r + \delta; H)$. Both are functions of the interest rate; equilibrium occurs at r^* such that $K(r^*) = A(r^*)$. We know the following facts about these functions:

- As $r \to -\delta$, $K(r) \to \infty$, since the required return on capital, $r + \delta$ is going to 0.
- As $r \to \infty$, $K(r) \to 0$, since the required return on capital is infinite.
- As $r \to \frac{1}{\beta} - 1$, the income fluctuation problem tells us that each individual's assets are unbounded, so $A(r) \to \infty$.
- As $r \to -1$, all loans and savings disappear, and $A(r)$ will go to the ad hoc borrowing constraint.

Under reasonable assumptions, $K(r)$ is continuous and monotonic, since it can be written as $F_K^{-1}(r + \delta; H)$. If $A(r)$ is continuous, then an equilibrium must exist, since the two curves must intersect. If $A(r)$ is also monotonic, then the steady state is unique.

To show that $A(r)$ is continuous and monotonic, we must consider λ^*:

- First, we must show that λ^* is unique using the theorem from measure theory:
  - *Compactness*: If $(1 + r)\beta < 1$ and preferences are CRRA, then the income fluctuation problem shows that there is always an upper bound, $\overline{a}$, on the asset space. The lower bound is provided by the borrowing constraint, and the state space of any Markov chain is compact.
  - *Feller property:* a' is bounded (since the asset space is bounded) and continuous (by the Theorem of the Maximum). This implies that $Q((a,\epsilon), \mathcal{A} \times \mathcal{E}) = \sum_{\epsilon' \in \mathcal{E}} \pi(\epsilon'|\epsilon) 1(a'(a,\epsilon) \in \mathcal{A})$ maps the set of bounded and continuous functions to itself.
  - *Monotonicty*: Note that a' is increasing. In general, a sufficient condition for monotonicity is positive autocorrelation in ϵ.

- *Monotone Mixing Condition*: Suppose the agent starts at $\overline{a}$. Then, for any $\hat{a}$, there is some K such that K consecutive shocks that are ϵ_1 will lead to an asset choice in $[-b, \hat{a}]$, because of consumption smoothing. Similarly, starting at $-b$ there is some K such that receiving K shocks equal to ϵ_N will lead to the range $[\hat{a}, \overline{a}]$. Since the sequences $(\epsilon_1, \epsilon_1, ..., \epsilon_1)$ and $(\epsilon_N, \epsilon_N, ..., \epsilon_N)$ have positive probabilities of occurring, this shows that the MMC holds.

- Then, we must show that λ^* is continuous with respect to r. Combining this with the fact that $a'(a, \epsilon; r)$ is continuous is r implies that $A(r)$ is continuous as well.

- We then wish to show that $A(r)$ is monotonic. There is no general proof of this, but it can be verified numerically. (Monotonicity implies that the substitution effect always dominates the income effect in $[-b, \overline{a}]$.)

Algorithm: Computing the Steady State

1. Choose $r^0 \in (-\delta, \frac{1}{\beta} - 1)$ to be the initial guess of the interest rate.

2. Given r^0 (and, implicitly, H, since it is fixed), obtain w^0 and K^0 from the production function, using the system of equations:

$$\begin{aligned} F_H(K^0, H) &= w^0 \\ F_K(K^0, H) &= r^0 + \delta \end{aligned}$$

3. Solve the household problem given the prices, r^0, w^0, using policy function iteration. This will yield the policy functions, $c(a, \epsilon; r^0)$ and $a'(a, \epsilon; r^0)$.

4. Given $\pi(\epsilon'|\epsilon)$ and $a'(a, \epsilon; r^0)$, compute $Q(r^0)$ and then $\lambda^*(r^0)$ using simulation:

   - Simulate histories of $a'(a_t, \epsilon_t)$ for M individuals, starting at any (a_0, ϵ_0) and using π, a' to update their states each period, t.
   - At each t, compute a vector, J_t, of summary statistics (mean, standard deviation, Gini coefficient, percentiles) of the wealth distribution of the M people.
   - When $\|J_t - J_{t-1}\|$ is sufficiently small, declare that the wealth distribution has converged and use the computed distribution of (a_t, ϵ_t) to approximate the distribution, λ^*.

5. Compute $A(r^0) = \int_{A \times E} a'(a, \epsilon; r^0) d\lambda^*(a, \epsilon; r^0)$ as the mean asset holdings from the last step of the simulation.

6. Compare $A(r^0)$ and $K(r^0)$. If they are equal, r^0 and the other computed quantities define the steady state. If $A(r^0) > K(r^0)$, then the next r^0 should be lower; otherwise, the next r^0 should be higher. One way to choose the next r^0 is to compute $F_K(A(r^0), H) - \delta$ and average the resulting interest rate with r^0.

To compute the steady state with endogenous labor choice, we must choose both r^0 and H^0 and adjust both until there is convergence.

This model explains wealth inequality by past earnings shocks and the resulting optimal savings behavior. Since earnings are exogenous, there is no theory of earnings inequality. Instead, luck determines wealth. To extend the model for additional inequality, agents could have different innate ability levels, so with the means, autocorrelations, and standard deviations of their income processes depends on the ability level. This comes only from initial heterogeneity. Empirically (from the Survey of Consumer Finances), there is more inequality in wealth than in earnings. The results of these models agree with that. However, the usual model generates too low a concentration of wealth for the rich and too much wealth accumulation for the poor. Ways to fix this include:

- Modifying the income process: Add a very high income realization that can occur with very small probability or add a model of entrepreneurship, in which there are increasing returns to good ideas, but this a is a more risky source of income.
- Introducing additional incentives for capital accumulation among the rich: bequest motives.
- Reducing incentives for capital accumulation among the poor: adding means-tested benefits for the poor or bankruptcy laws (which actually make the market more complete for the poor).
- Stochastic β or a bliss point in consumption.

4.1.1 Precautionary savings rate

Definition The *aggregate savings rate* is $s = \frac{I}{Y}$; in equilibrium, when $I = \delta K$, this is $s = \frac{\delta K}{Y}$. The *individual savings rate* is $s(a, \epsilon) = \frac{y - c(a,\epsilon)}{y}$ where $y = ra + w\epsilon$.

To quantify precautionary wealth, we note that under complete markets, $r = \frac{1}{\beta} - 1$, which implies the complete market capital stock. Then, $K(r^*) - K(\frac{1}{\beta} - 1)$ is the excess capital stock associated with precautionary savings. Alternatively, we could compute the difference in aggregate equilibrium savings rate. For a Cobb-Douglas production function, $Y = K^\alpha H^{1-\alpha}$, this has the closed-form expression:

$$\begin{aligned} r + \delta &= F_K(K, H) = \alpha \frac{Y}{K} \\ &= \alpha\delta \frac{Y}{\delta K} = \frac{\alpha\delta}{s} \\ s &= \frac{\alpha\delta}{r + \delta} \end{aligned}$$

This allows us to compute the savings rate at the complete and incomplete markets equilibria. The difference is the savings rates is called the *precautionary savings rate*. If the uncertainty increases, then the capital stock and aggregate savings rate increase, which will increase the level of output, but will lead to a consumption decline in order to do this; during the transition to a steady state with increased uncertainty, the economy would grow, but consumption would fall.

4.1.2 Comparative Statics

If the variance of income shocks increases, F_K is unchanged at each K, but $A(r)$ increases at each interest rate. This means that the equilibrium interest rate is lower and the capital stock is higher. An increase in risk aversion has the same result.

If an ad hoc borrowing limit increases, then in times of bad shocks, there is more slack and the agent can borrow more if needed. This shifts the $A(r)$ curve to the left. This leads to a result that is closer to complete markets. However, the increased potential borrowing will increase the interest rate, which will mitigate the amount of new borrowing. As the borrowing constraint approaches the natural borrowing limit, the constraint becomes less important.

To calibrate parameters to test the size of effects:

- The CRRA coefficient is usually considered to be between 1 and 3.
- The autocorrelation, ρ, in ϵ_t can be estimated from individual-level panel data on earnings (not in total income); the autocorrelation usually depends on the frequency of the data.
- β is usually chosen to match $\frac{K}{Y}$. If K is based only on financial wealth, then $\frac{K}{Y} \approx 2.5$. If K includes both financial wealth and housing wealth, then $\frac{K}{Y} \approx 4$. If β is higher, then agents are more patient and save more, so that $\frac{K}{Y}$ is higher as well.
- Empirically, for the Cobb-Douglas production function, $\alpha \approx 0.3$. Empirically, $\delta \approx 0.05\%$.
- It is hard to choose which interest rate the result of a calibration should match.

General empirical results include:

- With log utility and IID shocks, precautionary savings are almost 0, since there is low risk aversion and shocks are not persistent.
- With the CRRA parameter equal to 5 and $\rho = 0.9$, the precautionary savings rate is about 14%, since there is more risk aversion and more persistence in shocks. If the CRRA parameter is only 2, then the precautionary savings rate drops to 5%.

4.1.3 Optimal Taxation

With incomplete markets, there is a role for government taxation in order to provide insurance. Suppose the government taxes labor and transfers it lump sum to agents, so that the budget constraint is:

$$c + a' = Ra + (1-\tau)w\epsilon + t$$

If labor supply is inelastic, we may set $\tau = 1$ with no distortions, which leads to perfect insurance.

In order to add a labor supply decision, preferences become $E_0(\sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t))$, where h_t is hours worked. Then, the budget constraint is:

$$c + a' = Ra + (1-\tau)wh\epsilon + t$$

Now, the agent's decision consists of three policy functions, $c(a,\epsilon), a'(a,\epsilon), h(a,\epsilon)$. The equilibrium labor supply must satisfy:

$$H = \int_{A\times E} \epsilon h(a,\epsilon) d\lambda^*(a,\epsilon)$$

while the government budget constraint is:

$$t = \int_{A\times E} \tau w h(a,\epsilon)\epsilon d\lambda^*(a,\epsilon) = \tau w H$$

We define the welfare function, indexed by τ, as:

$$w(\tau) = \int_{A\times E} u\left(c^*(a,\epsilon;\tau), 1-h^*(a,\epsilon;\tau)\right) d\lambda^*(a,\epsilon;\tau)$$

(since the integral of utility over all agents is constant over time in the steady state). We could then find the Ramsey equilibrium as the optimal choice of τ^*. If $\tau < \tau^*$, then there is too little insurance; if $\tau > \tau^*$, then there is too much distortion.

This assumes the simplest form of tax policy. Extensions could include progressive labor tax rates, $T(wh)$, or lump sum transfers that go only to the poor.

4.2 Transitional Dynamics

A policy change will not only change the steady state but will have welfare effects along the transition as well.

When we are not in the steady state, the **household problem** is:

$$v_t(a,\epsilon) = \max_{c_t, a_{t+1}} \left(u(c_t(a,\epsilon)) + \beta \sum_{\epsilon_{t+1}\in E} v_{t+1}(a_{t+1}(a,\epsilon), \epsilon_{t+1}) \pi(\epsilon_{t+1}|\epsilon) \right)$$

subject to:

$$c_t(a,\epsilon) + a_{t+1}(a,\epsilon) \quad = \quad R_t a + w_t \epsilon (1-\tau_t) + \phi_t$$

where τ_t is the labor tax rate in period t and ϕ_t is the lump sum transfer. We assume that agents take τ_t as given and that ϕ_t adjusts to balance the government budget:

$$\phi_t \quad = \quad \int_{A\times E} \tau_t w_t h_t(a,\epsilon)\epsilon d\lambda_t(a,\epsilon) = \tau_t w_t H_t$$

The value function depends on time because the equilibrium is not stationary and therefore taxes, wages, and interest rates vary over time.

Definition Given an initial distribution, λ_0, and a tax sequence, $\{\tau_t\}_{t=1}^{\infty}$, a *competitive equilibrium* consists of:

- a sequence of value functions, $\{v_t(a,\epsilon)\}_{t=1}^{\infty}$ and associated optimal decision rules, $\{c_t(a,\epsilon), a_{t+1}(a,\epsilon)\}_{t=1}^{\infty}$,
- a sequence of firm choices of labor and capital, $\{K_t, H_t\}_{t=1}^{\infty}$,
- a sequence of prices, $\{w_t, R_t\}_{t=1}^{\infty}$,
- a sequence of transfers, $\{\phi_t\}_{t=1}^{\infty}$, and
- a sequence of distributions, $\{\lambda_t\}_{t=1}^{\infty}$,

such that, for all t,

- given prices, taxes and transfers, w_t, R_t, ϕ_t, τ_t, the value function, v_t, solves the household problem with associated policy functions, c_t, a_{t+1},
- given prices, w_t, R_t, firms are optimizing, so that $w_t = F_H(K_t, H_t)$ and $R_t + \delta = F_K(K_t, H_t)$,
- the labor market clears:

  $$H_t = \int_{A\times E} \epsilon h_t(a,\epsilon) d\lambda_t(a,\epsilon)$$

  (if λ_0 is already a steady state and the labor supply is inelastic, this is constant),
- the asset market clears:

  $$K_{t+1} = \int_{A\times E} a_{t+1}(a,\epsilon) d\lambda_t(a,\epsilon)$$

- the goods market clears:

$$\int_{A\times E} c_t(a,\epsilon)d\lambda_t(a,\epsilon) + (K_{t+1} - (1-\delta)K_t) = F(K_t, H_t)$$

- the government budget constraint holds:

$$\tau_t w_t H_t = \phi_t$$

- for all $\mathcal{A}\times\mathcal{E} \in B(A\times E)$, $\{\lambda_t\}_{t=0}^{\infty}$ satisfies:

$$\begin{aligned} \lambda_{t+1}(\mathcal{A}\times\mathcal{E}) &= \int_{A\times E} Q_t((a,\epsilon),\mathcal{A}\times\mathcal{E})d\lambda_t(a,\epsilon) \\ Q_t((a,\epsilon),\mathcal{A}\times\mathcal{E}) &= \sum_{\epsilon_{t+1}\in\mathcal{E}} 1(a_{t+1}(a,\epsilon)\in\mathcal{A})\pi(\epsilon_{t+1},\epsilon) \end{aligned}$$

This model assumes that the transition in taxes is completely unexpected up to time t and then is known with certainty after that.

Algorithm: Computing the Transition

1. For the initial tax rate, τ_0, and the final tax rate, τ_∞, compute the steady states, along with the corresponding household and firm functions, v_0, c_0, a_0', K_0 and $v_\infty, c_\infty, a_\infty', K_\infty$, using the previous algorithm.

2. Fix T at an arbitrary large number (such as 200), for which it is plausible that the economy will be at the new steady state in T periods.

3. Guess a sequence of aggregate capital stock, $\{\hat{K}_t\}_{t=0}^{T}$, such that $\hat{K}_0 = K_0$ and $\hat{K}_T = K_\infty$. This choice, together with H, implies paths for wages, interest rates, and transfers:

$$\begin{aligned} \hat{w}_t &= F_H(\hat{K}_t, H_t) \\ \hat{R}_t + \delta &= F_K(\hat{K}_t, H_t) \\ \hat{\phi}_t &= \hat{w}_t\tau_t H_t \end{aligned}$$

4. By assumption, $\hat{v}_T(a,\epsilon) = v_\infty(a,\epsilon)$. Use backward induction to compute the policy functions, $\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}'$, for $t = T-1, ..., 0$.

5. Given the sequence of decision rules, compute $\hat{Q}_t$ for each t, based on π and a_{t+1}. Use this to obtain $\{\hat{\lambda}_t\}_{t=0}^{T}$ where $\hat{\lambda}_0 = \lambda_0$.

6. Check that the asset market clears by computing $\hat{A}_{t+1} = \int_{A\times E} \hat{a}_{t+1}(a,\epsilon)d\hat{\lambda}_t(a,\epsilon)$. If $\sup_t |\hat{A}_t - \hat{K}_t| < \eta$, for a small η, then the sequence has converged. Otherwise, update the guess for $\hat{K}_t$, using $\hat{A}_t$, and return to step 3.

7. Check that $\hat{A}_T = K_\infty$. If not, return to step 2, guess a larger T and try again.

If the labor supply decision is endogenous, then we must guess a path of $\{H_t\}_{t=0}^{\infty}$ as well as a path for the capital stock.

Because we generally cannot prove that the steady state exists or is unique or that the transition path is unique, we should follow this with a stability analysis; one way is to start with a different guess for the path and see if the result is the same. The algorithm will have problems if the steady state or points along the path are unstable.

We may compute the welfare effect of the transition by comparing

$$v^*(a, \epsilon) = E_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_{t,0}) | a_0 = a, \epsilon_0 = \epsilon \right)$$

which is the expected discounted utility of living in the initial steady state forever, to

$$\tilde{v}(a, \epsilon) = E_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_t) | a_0 = a, \epsilon_0 = \epsilon \right)$$

which is the expected discounted utility along the transition.

Definition For a particular a, ϵ, choose $\omega(a, \epsilon)$ so that

$$E_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \right) = E_0 \left(\sum_{t=0}^{\infty} \beta^t u((1 + \omega(a, \epsilon))c_{t,0}) \right)$$

We define $\omega(a, \epsilon)$ to be the *conditional welfare change* associated with the transition. It is the percentage change in initial steady state consumption required to make the agent indifference between living in the original (rescaled) steady state and living through the transition to the new steady state. Note that the conditional welfare change is conditional on the initial values, a, ϵ. In the case of power utility, $u(c) = c^{-\sigma}$, we can find that:

$$\omega(a, \epsilon) = \left(\frac{\tilde{v}(a, \epsilon)}{v^*(a, \epsilon)} \right)^{1/\sigma} - 1$$

since we may factor $(1 + \omega(a, \epsilon))^{\sigma}$ out of the expectation.

Definition The *ex ante welfare* change, $\overline{\omega}$, is the change in consumption that would be necessary if all agents begin with $a = a_0$ and have ϵ_0 coming from the stationary distribution, $\pi^*(\epsilon)$. In general, this is given by the formula:

$$\sum_{i=1}^{N} E_0 \left(\sum_{t=0}^{\infty} \beta^t u\left((1 + \overline{\omega})c_{t,0}\right) |a_0 = a_0, \epsilon_0 = \epsilon_i \right) \pi^*(\epsilon_i) = \sum_{i=1}^{N} E_0 \left(\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) |a_0 = a_0, \epsilon_0 = \epsilon_i \right) \pi^*(\epsilon_i)$$

In the case of power utility, this is given by

$$\overline{\omega}(a, \epsilon) = \left(\frac{\sum_{\epsilon_i \in E} \tilde{v}(a, \epsilon_i)\pi^*(\epsilon_i)}{\sum_{\epsilon_i \in E} v^*(a, \epsilon)\pi^*(\epsilon_i)} \right)^{1/\sigma} - 1$$

This gives a single number for all agents, but depends on a_0.

Definition The *utilitarian social welfare function* is the change in welfare assuming that all agents have equal weights and are distributed according to the initial stationary distribution, λ^*. In the case of power utility, this is given by:

$$\overline{\overline{\omega}}(a, \epsilon) = \left(\frac{\int_{A \times E} \tilde{v}(a, \epsilon) d\lambda^*(a, \epsilon)}{\int_{A \times E} v^*(a, \epsilon) d\lambda^*(a, \epsilon)} \right)^{1/\sigma} - 1$$

The conditional welfare function gives the most complete description of what happens to agents; it can be used to see if the majority is better off, for example. However, the other measures are useful summaries.

With power utility, ex ante welfare can be decomposed into level and volatility effects:

$$\overline{\omega} = (1 + \omega^{LEV})(1 + \omega^{VOL})$$

The volatility effect measures how the policy affects the volatility of consumption; if insurance increases, then the consumption will be less volatile, which is a positive volatility effect. The level effect measures how the policy affects average consumption. If insurance increases, the savings and therefore capital will decrease, so that the effect on consumption is ambiguous. If the labor supply is endogenous, the level effect is likely to be negative when the tax increases.

4.3 Aggregate Uncertainty

One model with aggregate uncertainty is given by:

- *Aggregate uncertainty* is provided by an aggegrate TFP shock, so that $Y_t = Z_t F(K_t, H_t)$, where Z_t follows a two state Markov chain with $Z_b < Z_g$. The aggregate state space is given by $(z, \lambda) \in Z \times \Lambda$, where λ is the distribution of agents of the individual state space.

- *Individual level uncertainty* is given by ϵ_t, which also follows a two-state Markov chain, with $0 = \epsilon_b < \epsilon_g$; we consider ϵ_b to be unemployment. The individual state space is given by $(a, \epsilon) \in A \times E$.

- The two types of uncertainty follow a joint Markov chain; this allows the probability of unemployment to vary with the overall state of the economy.

The **household problem** is given by:

$$\begin{aligned}
v(a,\epsilon;z,\lambda) &= \max_{c,a'}\left(u(c)+\beta\sum_{\epsilon'\in E,z\in Z} v(a',\epsilon';z',\lambda')\pi(\epsilon',z'|\epsilon,z)\right)\\
c+a' &= w(z,K)\epsilon + r(z,K)a\\
a' &> 0\\
K &= \int_{A\times E} a\,d\lambda(a,\epsilon)\\
\lambda' &= G(z,\lambda,z')
\end{aligned}$$

When we model aggregate risk, we must include $\lambda(a,\epsilon)$ as part of the state of the problem, since it determines all of the aggregate variables and changes each period. Note that the agent needs to know λ in order to compute K' next period (by way of computing λ').

Definition A *recursive competitive equilibrium with aggregate uncertainty* consists of

- a value function, v, and policy functions, a', c,
- firm policies, H, K,
- prices, r, w, and
- a law of motion, G,

such that

- given pricing functions, $r(z,K)$,$w(z,K)$, the policy functions, a', c solve the household's problem given by the value function, v,
- given $r(z,K)$ and $w(z,K)$, the firm optimally chooses K, H, so that $r(z,K)+\delta = zF_K(K,H)$ and $w(z,K) = zF_H(H,K)$,
- the labor market clears: $H = \int_{A\times E}\epsilon d\lambda$,
- the asset market clears: $K = \int_{A\times E} a\,d\lambda$,
- the goods market clears:

$$\int_{A\times E}\left(c(a,\epsilon;z,\lambda)+a'(a,\epsilon;z,\lambda)\right)d\lambda = zF(K,H)+(1-\delta)K$$

- the aggregate law of motion is generated by the Markov chain, π, and the policy function, a':

$$\begin{aligned}
\lambda'(\mathcal{A}\times\mathcal{E}) &= G(z,\lambda,z') = \int_{A\times E} Q_{z,z'}((a,\epsilon),\mathcal{A}\times\mathcal{E})d\lambda(a,\epsilon) \\
Q_{z,z'}((a,\epsilon),\mathcal{A}\times\mathcal{E}) &= \sum_{\epsilon\in\mathcal{E}} 1\left(a'(a,\epsilon;z,\lambda)\in\mathcal{A}\right)\pi(z',\epsilon|z,\epsilon)
\end{aligned}$$

Including λ as part of the state is a problem, since it is an infinite-dimensional object. As an alternative, we look for a finite-dimensional approximation (which suggests that agents have partial information or bounded rationality) and show that this approximation is good enough. It would be enough if agents could find a law of motion for K directly, so that they could form expectations of prices next period.

Let m be an $n\times 1$ vector of moments of λ. Then, an approximation of the aggregate state is given by $(z,m_1,...,m_n)$. As $n\to\infty$, the representation of λ is exact; if n is finite and agents use this approximation, then they have partial information or bounded rationality. In this case, it is sufficient to specify a law of motion for m, $m'=G_n(z,m,z')$; since agents decide on their next-period assets before z' is known, z' affects only the distribution of ϵ in the next period. Then, we use m to approximate K. One functional form is:

$$\ln K' = b_{z0} + b_{z1}\ln K + b_{z2}(\ln K)^2 + ...$$

which is a log-linear representation in which the coefficients depend on the current aggregate state, z; empirically, the first two terms are sufficient. This reduces the agents' state variables to $(a,\epsilon;z,K)$ with a simple law of motion, which reduces the household problem to:

$$\begin{aligned}
v(a,\epsilon;z,K) &= \max_{c,a'}\left(u(c)+\beta\sum_{\epsilon'\in E,z\in Z} v(a',\epsilon';z',K')\pi(\epsilon',z'|\epsilon,z)\right) \\
c+a' &= w(z,K)\epsilon + r(z,K)a \\
a' &> 0 \\
\ln K' &= b_{z0} + b_{z1}\ln K
\end{aligned}$$

Definition The *aggregate consistency condition* is that, in equilibrium, the agents' law of motion for the aggregate variables leads to decisions that create the same aggregate law of motion.

Once we have found an equilibrium for an equilibrium with a certain approximation, we may add a term and recalculate to check if there are sizeable gains in utility for using a better approximation.

Algorithm: Finding an Approximate Equilibrium

1. Guess b_{z0}, b_{z1}.
2. Solve the household problem with these coefficients to find $a'(a, \epsilon; z, K)$ and $c(a, \epsilon; z, K)$.
3. Simulate histories of length T for I individuals to create a wide, long artificial panel, keeping track of asset holdings in each period.
4. At each t, compute $K_t = \frac{1}{I}\sum_{i=1}^{I} a_t^i$ to find the sequence $\{K_t\}_{t=1}^{T}$.
5. Estimate the coefficients on the implied law of motion with OLS:

$$\ln K_{t+1} = \beta_{z0} + \beta_{z1} \ln K_t$$

   for each z.
6. Check if b_{z0}, b_{z1} is close enough to β_{z0}, β_{z1} for each z. If not, return to step 1 with $b_{z0} = \beta_{z0}$ and $b_{z1} = \beta_{z1}$ (or something between the two of them).
7. After convergence, check the success of the approximation by looking at R^2 from the regression.
8. Add another term to the approximation and return to Step 1, to see how much the second term improves the approximation; stop when there is little improvement in R^2.

Alternative approximation methods include other functional forms, finding law of motion for R, w directly, and assuming a particular parametric form for λ with unknown parameters. Empirically (Krusell and Smith), the linear term leads to $R^2 = 0.999998$, which is good enough.

This provides a *near-aggregation result.* Suppose that the asset policy function is of the form:

$$a'(a, \epsilon; z, \lambda) \approx b_{z0} + b_{z1}a + b_{z2}\epsilon$$

Then, we find that:

$$\begin{aligned} K' &= \int_{A\times E} a'(a, \epsilon; z, \lambda) d\lambda(a, \epsilon) \\ &= \int_{A\times E} (b_{z0} + b_{z1}a + b_{z2}\epsilon) d\lambda(a, \epsilon) \\ &= b_{z0} + b_{z1}K + b_{z2}H \end{aligned}$$

Since H is constant once z is known, this gives the law of motion $K' = \tilde{b}_{0z} + b_{1z}K$. Since the law of motion in terms of logs can be approximated by a linear function, this shows that the decision rule is approximately linear in a, ϵ, so that there is near-aggregation.

This also shows that the policy function is very linear, except near the borrowing constraint. The curvature near the borrowing constraint is not very important because, first, not many people are near the extremes (λ has thin tails) and, second, because the people with very little capital have less effect on the aggregate capital. Thus, we have near aggregation because the policy function is linear in the area that matters most for the aggregation. This shows that market incompleteness is not very important for the evolution of aggregate variables, Y, K, H, C, and that the representative agent model is robust to some market incompleteness. However, the inequality in the model would still matter for the agents.

Near-aggregation continues to hold with more shocks. However, aggregates such as asset prices and the welfare cost of business cycles will be more affected by market structure.

4.4 Constrained Efficiency in Aiyagari Models

We usually compare the outcomes with incomplete markets to the outcome with a complete market. However, we can also compare them to the result if a social planner was constrained to the same asset structure, which may give a different result.

Definition *Constrained efficiency* is measured by comparing the social planner's decision rules for households to their own decision rules, when the available assets are identical.

The competitive equilibrium decision rule depends on the prices in the stationary equilibrium, which in turn depend on the steady state distribution in that equilibrium; the distribution and therefore the prices may change in the social planner allocation. That is, the agent's Euler equation can be written more completely as:

$$u'(R(\lambda^*)a + w(\lambda^*)\epsilon - a'(a,\epsilon)) \geq R\beta \sum_{\epsilon' \in E} \pi(\epsilon', \epsilon) u'\left(R(\lambda^*) + w(\lambda^*)\epsilon' - a'(a'(a,\epsilon), \epsilon')\right)$$

The *constrained planner problem* is given by the planner value function and constraints:

$$\begin{aligned}
\Omega(\tilde{\lambda}) &= \max_{\tilde{a}(a,\epsilon)} \left(\int_{A \times E} u\left(R(\tilde{\lambda})a + w(\tilde{\lambda})\epsilon - \tilde{a}(a,\epsilon)\right) d\tilde{\lambda}(a,\epsilon) + \beta\Omega(\tilde{\lambda}) \right) \\
\tilde{\lambda}(\mathcal{A} \times \mathcal{E}) &= \int_{A \times E} 1\left(\tilde{a}(a,\epsilon) \in \mathcal{A}\right) \pi(\epsilon'|\epsilon) d\tilde{\lambda}(a,\epsilon) \\
R(\tilde{\lambda}) &= F_K(K,H) - \delta \\
w(\tilde{\lambda}) &= F_H(K,H) \\
K &= \int_{A \times E} \tilde{a}(a,\epsilon) d\tilde{\lambda}(a,\epsilon) \\
H &= \int_{A \times E} \epsilon d\tilde{\lambda}(a,\epsilon)
\end{aligned}$$

The welfare theorems no longer hold, so the equilibrium and the constrained planner solution need not agree. Instead, incomplete markets lead to a sort of externality, where the agents' actions have effects that are not reflected in prices (in this case, it is their effect on aggregate capital).

The first order condition for the social planner is:

$$u'(c) \geq R(\tilde{\lambda})\beta \sum_{\epsilon' \in E} \pi(\epsilon'|\epsilon)u'(c') + \int_{A \times E} (\epsilon' F_{HK} + a' F_{KK})u'(c')d\tilde{\lambda}$$

The first term on the right-hand side corresponds to the agent's Euler equation. The second term is the effect of saving an additional unit of capital on income, through wages, $\epsilon' F_{HK}$, and interest rates, $a' F_{KK}$, weighted by marginal utility and integrated over all agents. This shows that the planner can use price manipulation as a transfer to the poor. If the poor have labor income but little capital income, then $\epsilon' F_{HK} > 0$ and $a' F_{KK} \approx 0$, since they have the largest $u'(c')$, the second term will be larger. Then every agent should save more. This leads to more capital and therefore higher wages, which redistributes income to the poor.

Compared to the constrained efficient allocation, the competitive equilibrium has too little savings; compared to the complete markets equilibrium, people are saving too much. Thus the belief about whether people should save more or less (and policy consequences of that belief) depends on the choice of benchmark.

Depending on the types of shocks, the sign of the second term may change (for example, if the poor are living off of savings only, because they are unemployed).

5 Beyond the Bond Economy

5.1 Limited Enforcement Models

Suppose we have an Arrow-Debreu economy with time 0 trading. We now assume that if an individual fails to carry out his promised trades, then he must live in autarky forever after. Then, in equilibrium, contracts will be made only if the value of the contract at that time exceeds the value of default. This limits the contracts that are made, but eliminates default.

If there were complete markets, marginal utility and value functions would be equal across all agents in all periods:

$$\begin{aligned} c(h_t) &= c^{CM} = \sum_{s=1}^{S} \pi_s \overline{y}_s \\ V^{CM} &= u(c^{CM}) + \beta V^{CM} \\ &= \frac{1}{1-\beta} u(c^{CM}) \end{aligned}$$

That is, each agent consumes the average income. If there were autarky, each individual would consume his own endowment, so that:

$$
\begin{aligned}
c(h_t) &= y_t \\
V^{AUT} &= \frac{1}{1-\beta}E(u(y_t)) \\
&= \frac{1}{1-\beta}\sum_{s=1}^{S} u(y_s)\pi_s \\
&< V^{CM}
\end{aligned}
$$

(The last inequality follows from Jensen's inequality.)

One model is the *village/money-lender* model:

- There is a continuum of infinitely-lived *households* of measure one, with discount rate β, preferences ordered by $E_0(\sum_{t=0}^{\infty}\beta^t u(c_t))$, with u strictly increasing and strictly concave.

- Each household is subject to independent and identically distributed *endowment shocks*, $y_t \in \{\overline{y}_1, ..., \overline{y}_S\}$, with probabilities given by $\pi_1, ..., \pi_S$. We let $h_t = (y_0, ..., y_t)$ be an individual's history of shocks.

- There is a unique individual called the *money-lender*, who has access to markets and storage at rate $R = \frac{1}{\beta}$. This individual is risk-neutral.

- All contracts between individuals and the money lender have *one-sided commitment*, in which the money-lender commits but a household can default and then be barred from any future trading.

- The contract is agreed on at time 0 and given by $K_0 = \{c_t\}_{t=0}^{\infty}$, where $c_t = f_t(h_t)$. That is, the household gives y_t to the money-lender at each time and consumes c_t which may depend on the previous payments. The value of the contract for the household is:

$$V(K_0) = E_0\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right)$$

  while the value (expected profits) for the money-lender is:

$$P(K_0) = E_0\left(\sum_{t=0}^{\infty}\beta^t (y_t - c_t)\right)$$

  Because of the differing utility functions, the agent is willing to pay more in order to smooth consumption, and the money-lender makes a profit.

- The *money-lender's problem* is to maximize profits while giving enough utility to the agent so that the agent won't default:

$$\begin{aligned} P^*(V_0) &= \max_{K_0} P(K_0) \\ V(K_0) &\geq V_0 \end{aligned}$$

  where V_0 is the minimum value the agent must receive in order not to default; in this case, $V_0 = A^{AUT}$.

Since the profit is decreasing in the agent's utility, the constraint will hold with equality. Let $K_{0/1}$ be the period 1 continuation of the contract K_0 chosen at time 0. Then,

$$\begin{aligned} P(K_0) &= E_0\left((y_0 - c_0) + \beta E_1\left(\sum_{t=0}^{\infty} \beta^j (y_{j+1} - c_{j+1})\right)\right) \\ &= E_0\left((y_0 - c_0) + \beta P(K_{0/1})\right) \\ V(K_0) &= E_0\left(u(c_0) + \beta E_1\left(\sum_{j=0}^{\infty} \beta^j u(c_{j+1})\right)\right) \\ &= E_0\left(u(c_0) + \beta V(K_{0/1})\right) \end{aligned}$$

Then, the money-lender's problem is:

$$\begin{aligned} P*(V_0) &= \max_{\{c_0, K_{0/1}\}} E_0\left((y_0 - c_0) + \beta P(K_{0/1})\right) \\ E_0\left(u(c_0) + \beta V(K_{0/1})\right) &= V_0 \end{aligned}$$

Let $(c_0^*, K_{0/1}^*)$ solve the money-lender's problem. Let $\omega = V(K_{0/1}^*)$. Consider the agent's problem at time 1:

$$\begin{aligned} P^*(\omega) &= \max_{K_1} P(K_1) \\ V(K_1) &\geq \omega \end{aligned}$$

We must have $P(K_{0/1}^*) \leq P^*(\omega)$, since the agent is fully optimized in the first period and therefore must do at least as well; that is, K_1^* was one of the possible choices for $K_{0/1}^*$. This means that we may separate the maximization operator:

$$\begin{aligned} P^*(V_0) &= \max_{c_0} E_0\left((y_0 - c_0) + \beta \max_{K_{0/1}} P(K_{0/1})\right) \\ E_0\left(u(c_0) + \beta V(K_{0/1})\right) &= V_0 \end{aligned}$$

In addition, $P(K_{0/1}) = P^*(\omega)$. (If we instead had $P^*(\omega) > P(K^*_{0/1})$, then the contract, K^*_1, chosen for $P^*(\omega)$ would be feasible for the money lender and would yield a bigger profit, which is contrary to our assumptions.) This yields the recursive formulation:

$$\begin{aligned} P^*(V_0) &= \max_{c_0} E_0\left((y_0 - c_0) + \beta P^*(\omega)\right) \\ E_0\left(u(c_0) + \beta\omega\right) &= V_0 \end{aligned}$$

This means that choosing the entire sequence of contingent payments is equivalent to choosing the level of continuation utility, ω, for the agent. The problem next period is:

$$\begin{aligned} P^*(\omega) &= \max_{c} E_0\left((y - c) + \beta P^*(\omega')\right) \\ E_0\left(u(c) + \beta\omega'\right) &= \omega \end{aligned}$$

The constraint is called the *promise-keeping constraint*, and yields a law of motion for the state variable, ω. This shows that there are two ways to deliver utility to the agent: current utility ($u(c)$) and future utility (ω'). Since the payoff depends on the current payment of the household, the money lender can give more total utility during good income shocks, to prevent default. However, the choices will still smooth consumption.

Combining this gives the recursive formulation of the money-lender's problem:

$$P(v) = \max_{\{c_s,\omega_s\}} \sum_{s=1}^{S} \left(\overline{y}_s - c_s + \beta P(\omega_s)\right) \pi_s$$

such that

$$\begin{aligned} \sum_{s=1}^{S} \left(u(c_s) + \beta\omega_s\right) \pi_s &= v \\ u(c_s) + \beta\omega_s &\geq u(\overline{y}_s) + \beta V^{AUT}, s = 1, ..., S \\ c_s &\in [c_{min}, c_{max}] \\ \omega_s &\in [\omega_{min}, \omega_{max}] \end{aligned}$$

Payments and continuation utilities are state-contingent, since they are chosen before y_s is known. We assume boundedness here, to ensure that the maximization is well-behaved. However, it is easy to see that $\omega_{min} = V^{AUT}$ and $\omega_{max} = \frac{1}{1-\beta}\overline{y}_S$ work as bounds.

The Lagrangian is:

$$L = \sum_{s=1}^{S} (\overline{y}_s - c_s + \beta P(\omega_s))\pi_s + \mu \left(\sum_{s=1}^{S} \left(u(c_s) + \beta\omega_s\right) \pi_s - v \right) + \sum_{s=1}^{S} \lambda_s \left(u(c_s) + \beta\omega_s - u(\overline{y}_s) - \beta V^{AUT}\right)$$

The first order conditions and envelope theorem yield:

$$\begin{aligned} -\pi_s + \mu u'(c_s)\pi_s + \lambda_s u'(c_s) &= 0 \\ \pi_s \beta P'(\omega_s) + \mu\beta\pi_s + \beta\lambda_s &= 0 \\ P'(v) &= -\mu \end{aligned}$$

Rearranging and solving shows that $\frac{u'(c_s)}{\beta} = -\frac{1}{\beta P'(\omega_s)}$. The left-hand-side is the marginal rate of substitution between today's utility and future utility, since the total utility is simple $u(c_t)+\beta\omega_t$. The right-hand-side is the marginal rate of transformation for the money-lender between c_t and ω_t, since consumption costs 1 and future utility costs $\beta P'(\omega_t)$.

We may also use the envelope theorem and first order conditions to find that $P'(\omega_s) = P'(v) - \frac{\lambda_s}{\pi_s}$. This yields two cases:

- *Case* $\lambda_s = 0$: In this case, the participation constraint does not bind, and $\omega_s = v$ by the strict concavity of P. Then, the first order conditions imply that $c_s = f_1(v)$, and the consumption level depends only on v, not on $\overline{y}_s$. This is full insurance, since the endowment shock does not affect consumption; a sequence of low values of the endowment shock lead to a sequence of identical (c, ω).

- *Case* $\lambda_s > 0$: In this case, the participation constrain binds, which means that $P'(\omega_s) < P'(v)$ and $\omega_s > v$ by concavity. This means that the money-lender must promise more future utility to the household. Differentiating the first order condition also shows that:

$$\begin{aligned} u''(c_s)dc_s &= P'(\omega_s)^{-2}P''(\omega_s)d\omega_s \\ \frac{dc_s}{d\omega_s} &= \frac{P''(\omega_s)}{P'(\omega_s)^2 u''(c_s)} > 0 \end{aligned}$$

  (The sign follows from the concavity of P and u.) This shows that consumption increases with ω_s. Furthermore, we may compute:

$$\begin{aligned} u(c_s) + \beta\omega_s &= u(\overline{y}_s) + \beta V^{AUT} \\ u'(c_s) &= -P'(\omega_s)^{-1} \end{aligned}$$

  This shows that c_s, ω_s do not depend on v but only on $\overline{y}_s$, so that $c_s = f_2(\overline{y}_s)$ and $\omega_s = g_s(\overline{y}_s)$. This is the "amnesia property of the optimal contract", because the previous history of the contract stops mattering once the participation constraint binds.

Combining these two cases shows that consumption is an increasing step function over time. If $y_T < \max(y_1, ..., y_{T-1})$, then consumption stays constant; otherwise, consumption and ω increase. After the first time that $y_T = \overline{y}_S$, consumption and ω are constant forever. However, the level is lower than consuming $\overline{y}_S$ forever. At the maximum level,

$$\omega_s = \frac{u(c_s)}{1-\beta} = \beta V^{AUT} + (1-\beta)\frac{u(\overline{y}_S)}{1-\beta}$$

which is the weighted average of autarky and getting the best consumption level forever.

In the limit, the distribution of consumption over households is degenerate (though it is non-trivial at any finite period). Some models have finite-lived agents where the new agents begin at the low level; this leads to a non-degenerate steady state distribution of consumption.

5.2 Other Models

One way to extend the standard incomplete markets model is to add assets that have appeared in reality. This includes models with equity markets, bankruptcy, unemployment insurance, families and networks, social security, annuities, life insurance, and government debt. However, this does not give an endogenous explanation of why such instruments appear.

Alternatively, one can endogenize the market structure and amount of insurance. Besides the imperfect enforcement model of the previous section, there might be *private information constraints*; this leads to the contract theory literature. Private information models, unlike limited enforcement models, lead to dispersion growing forever.