Macroeconomics Summary

Rebecca Sela

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In general, every model consists of preferences, endowments, technology, and demography.

When defining a competitive equilibrium, one should be sure to specify all of the conditions for household maximization, the government budget condition, and for market clearing.

1 Asset Pricing

In asset pricing, we generally use representative agent models, in which agents are identical or they can be aggregated. These models can either be partial equilibrium models using Euler equations or general equilibrium models using Lucas Trees.

1.1 Partial Equilibrium and No Arbitrage Pricing

Suppose prices are exogenous and the representative agent wishes to maximize $E_t(\sum_{j=0}^{\infty} \beta^j u(c_{t+j}))$, where c_t is consumption. The agent can choose wealth, A_t, in the form of *bonds*, L_t, which are claims to one unit of the good at time $t+1$ at an interest rate R_t, and a number of shares (N_t), each of which pays a dividend in each period. Let p_t be the share price and y_t be the dividend paid on the shares owned at the beginning of the period. Then, the consumer's Euler equations are:

$$\begin{aligned} \frac{u'(c_t)}{R_t} &= E_t(\beta u'(c_t)) \\ u'(c_t)p_t &= E_t(\beta(p_{t+1} + y_{t+1})u'(c_{t+1})) \end{aligned}$$

To solve asset pricing problems in full details:

- Write the maximization problem.

- Write the Bellman equation.
- Derive the Euler equation from the first order conditions and the envelope theorem.
- Compute asset prices from the social planner's allocation and the Euler equation.

For arbitrage-pricing arguments, one should know what to buy and sell in order to make money (whether one price is too high or too low). No-arbitrage arguments involve only prices, never probabilities.

No-arbitrage pricing can be based on:

- Computing the household budget constraint (possibly in multiple periods) to ensure that the budget set is bounded.
- Using the firm's zero profit conditions.
- Finding two prices and relating them to ensure that there is no way to make a sure profit. This is done by finding a redundant price (pinned down by all the others). For example, if there is only one asset, there can be only one intertemporal price.

1.2 Lucas Tree Economy

Suppose the representative agent wishes to maximize $E_t(\sum_{j=0}^{\infty} \beta^j u(c_{t+j}))$, where c_t is consumption. The agent can choose wealth, A_t, in the form of risk-free *bonds*, L_t, which are claims to one unit of the good at time $t+1$ at an interest rate R_t, and a number of shares ("trees"), each of which pays a non-storable dividend ("fruit") in each period. Let p_t be the share price and y_t be the dividend paid on the shares owned at the beginning of the period. Because this is general equilibrium, the price now depends on the current state. We assume that y_t is the function of a Markov state, s_t, on a continuous state space with:

$$F(s', s) = P(s_{t+1} \leq s' | s_t = s) = \int_{-\infty}^{s'} f(z, s) dz$$

Generally, we normalize that each agent has one tree ($N_t = 1$).

Then, the budget constraint is:

$$\begin{aligned} c_t + \frac{L_t}{R_t} + p_t N_t &\leq A_t \\ A_{t+1} &= L_t + (p_{t+1} + y_{t+1}) N_t \end{aligned}$$

We may also have borrowing constraints ($L_t \geq \overline{L}$ or $N_t \geq \overline{N}$) to ensure solvency.

The agents' Euler equations are:

$$\begin{aligned}\frac{u'(c_t)}{R_t} &= E_t(\beta u'(c_t)) \\ u'(c_t)p_t &= E_t(\beta(p_{t+1}+y_{t+1})u'(c_{t+1}))\end{aligned}$$

We also have the transversality conditions:

$$\begin{aligned}\lim_{k\to\infty} E_t\left(\beta^k u'(c_{t+k})\frac{L_{t+k}}{R_{t+k}}\right) &= 0 \\ \lim_{k\to\infty} E_t(\beta^k u'(c_{t+k})p_{t+k}N_{t+k}) &= 0\end{aligned}$$

These ensure that the wealth does not increase or decrease unboundedly in the limit, which pins down the level of consumption.

If $R_t = R > 1$ is constant, then $\frac{1}{\beta R}u'(c_t) = E_t(u'(c_{t+1}))$, and marginal utility follows a Markov process. If utility is quadratic, then marginal utility is linear and $c_{t+1} = \textit{constant} + \frac{1}{\beta R}c_t + \epsilon_{t+1}$.

In general, we may use the second Euler equation (and the fact that $E(AB) = E(A)E(B) + Cov(A,B)$) to write:

$$\begin{aligned}p_t &= E_t\left(\beta(p_{t+1}+y_{t+1})\frac{u'(c_{t+1})}{u'(c_t)}\right) \\ &= \beta E_t(p_{t+1}+y_{t+1})E_t\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) + \beta Cov_t\left(p_{t+1}+y_{t+1}, \frac{u'(c_{t+1})}{u'(c_t)}\right)\end{aligned}$$

The covariance term is zero if agents are risk neutral (so that marginal utility is constant) or there is no uncertainty.

We can solve the social planner's problem (with the representative agent) to find the equilibrium allocation (the representative agent consumes all the fruit in each period). Then, we use the equilibrium allocation to compute prices:

$$\begin{aligned}\frac{1}{R_t} &= \frac{E_t(\beta u'(y_{t+1}))}{u'(y_t)} \\ u'(y_t)p_t &= E_t(\beta(p_{t+1}+y_{t+1})u'(y_{t+1})) \\ &= E_t(\beta p_{t+1}u'(y_{t+1})) + E_t(\beta y_{t+1}u'(y_{t+1})) \\ &= \ldots \\ &= E_t\left(\sum_{j=1}^{\infty}\beta^j u'(y_{t+j})y_{t+j}\right) + E_t\left(\lim_{k\to\infty}\beta^k u'(y_{t+k})p_{t+k}\right)\end{aligned}$$

where the limit term is zero by the transversality condition (and the equilibrium condition that each agent continues to hold one tree). This yields the price of the asset as a function of the current state:

$$p_t = E_t \left(\sum_{j=1}^{\infty} \beta^j u'(y_{t+j}) y_{t+j} \right) \frac{1}{u'(y_t)}$$

This yields a stochastic process for prices that depends on the stochastic process for dividends.

Timing affects prices in this economy. We generally assume that one receives the dividends from trees bought last period; that is, trees are traded after dividends are paid. If this were not true, then prices and budget constraint would be:

$$\begin{aligned} p_t &= \beta E_t \left(\frac{u'(c_{t+1})}{u'(c_t)} \right) p_{t+1} + d_t \\ p_t N_t + c_t &\leq N_t y_t + N_{t-1} p_t \end{aligned}$$

However, the general conclusions would not change.

(This model is in contrast to the real business cycles model, which depends on capital and includes persistence in the technology shocks.)

1.3 Term Structure of Interest Rates

In general, to price assets, there are two options:

- Use the prices that are already known from the economy in a *no-arbitrage argument*.
- Add the new asset into the budget constraint and solve the general equilibrium model again to determine its price.

Definition 1 *The* term structure *describes the relationship between a risk-free bond's time to maturity and its per-period yield.*

Let R_{jt} be the risk-free gross return between t and $t + j$. Let L_{jt} be the quantity of these bonds. Then, the budget constraint becomes:

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \frac{L_{jt}}{R_{jt}} + p_t N_t &\leq A_t \\ A_{t+1} &= L_{1t} + \sum_{j=2}^{\infty} \frac{L_{jt}}{R_{j-1,t+1}} + (p_{t+1} + y_{t+1}) N_t \end{aligned}$$

where A_0 may be given and we impose some borrowing constraint. Since the dividends can be described by a Markov process, we have a value function:

$$V(A_t, s_t) = \max\{u(c_t) + \beta E_t(V(A_{t+1}, s_{t+1}))\}$$

where the maximization is taken over the constraints above. The Lagrangian yields the first order conditions with respect to c_t, L_{jt}:

$$\begin{aligned}
u'(c_t) - \lambda_t &= 0 \\
-\beta E_t\left(V_1(A_{t+1}, s_{t+1})\frac{\partial A_{t+1}}{\partial L_{jt}}\right) - \frac{\lambda_t}{R_{jt}} &= 0 \\
V_1(A_{t+1}, s_{t+1}) &= u'(c_{t+1}) \\
\frac{1}{R_{jt}} &= \beta E_t\left(\frac{u'(c_{t+1})}{u'(c_t)} \cdot \frac{1}{R_{j-1,t+1}}\right)
\end{aligned}$$

Since the social planner's problem yields an equilibrium allocation of $c_t = y_t$, this yields the term structure of interest rates:

$$\frac{1}{R_{jt}} = \beta E_t\left(\frac{u'(y_{t+1})}{u'(y_t)} \cdot \frac{1}{R_{j-1,t+1}}\right) = \beta^j E_t\left(\frac{u'(s_{t+j})}{u'(s_t)}\right)$$

In general, R_{jt} is a function of $y^t = (y_0, ..., y_t)$. If y_t is a Markov process, the interest rate term structure depends only on y_t, not on t or the history. We may also calculate the per-period interest rate:

$$\tilde{R}_{jt} = \frac{1}{\beta}\left(\frac{u'(s_t)}{E_t(u'(s_{t+j}))}\right)^{1/j}$$

Suppose s_t is independent and identically distributed. Then:

$$R_j(s_t) = \frac{1}{\beta^j}\frac{u'(s_t)}{E(u'(s))} = R_1(s_t)\frac{1}{\beta^{j-1}}$$

Definition 2 *In* pure expectations theory, $\frac{1}{R_{2t}} = \frac{1}{R_{1t}}E_t(\frac{1}{R_{1,t+1}})$*. In this case, there is no risk premium.*

In general, we may split the expectation of the product to compute:

$$\frac{1}{R_{2t}} = \beta E_t\left(\frac{u'(s_{t+1})}{u'(s_t)}\right)E_t\left(\frac{1}{R_{1,t+1}}\right) + \beta Cov_t\left(\frac{u'(s_{t+1})}{u'(s_t)}, \frac{1}{R}\right)$$

Pure expectations theory holds if the second term (the *risk premium*) is 0. This will occur if agents are risk neutral or if there is no uncertainty. For an IID process, the second term becomes $\beta^2 \frac{E(u'(s))}{u'(s_t)} Cov(u'(s_{t+1}), \frac{1}{u'(s_{t+1})})$, which is always negative. In this case, a two-period bond must pay more interest than a one-period bond because gaining knowledge of the second period might change decisions. (If there is bad news next period, then $u'(s_{t+1})$ will be higher, which will make the subsequent interest rate higher.) The expression above again reduces to $\frac{1}{R_{2t}} = \beta\frac{1}{R_{1t}}$.

(This covariance is negative even if agents are risk-loving.)

1.4 No-Arbitrage Pricing

Suppose $\{s_t\}$ is Markov. Then, the pricing kernel is:

$$Q_j(s_j, s) = \beta^j \frac{u'(s_j)}{u'(s)} f^j(s_j, s)$$

where $f^j(z, s) = \int f^1(z, s_{j-1}) f^{j-1}(s_{j-1}, s) ds_{j-1}$ is the j-step ahead transition function.

For example, consider an insurance policy that pays one good next period if $y_{t+1} \leq \alpha$. This could be added to the budget constraint to find the price. Instead, we use a no-arbitrage argument to say that $q_\alpha(s) = \int_{-\infty}^{\alpha} 1Q_1(s', s)ds'$. We may also express Q_1 in terms of primitives (though this goes beyond a no-arbitrage argument, since no-arbitrage arguments never depend on probabilities, only on prices):

$$\begin{aligned} q_\alpha(s) &= \int_{-\infty}^{\alpha} 1Q_1(s', s)ds' \\ &= \int_{-\infty}^{\alpha} \beta \frac{u'(s')}{u'(s)} f(s', s)ds' \\ &= E\left(\beta \frac{u'(s_{t+1})}{u'(s_t)} \middle| s_{t+1} \leq \alpha, s_t\right) P(s_{t+1} \leq \alpha | s_t) \end{aligned}$$

If agents are risk neutral, this reduces to $q_\alpha(s) = \beta P(s' \leq \alpha)$, since the ratio of marginal utilities is constant at 1. If agents are risk averse, $\frac{u'(s')}{u'(s)} > 1$ when $s' \leq \alpha \leq s$, so that individuals will pay more for insurance if the current period is better than α.

1.5 Modigliani and Miller Theorem

Suppose we wish to sell off a tree (or a firm) using B bonds that pay r each period (as long as $rB \leq Y_t$; otherwise, they pay $\frac{Y_t}{B}$) and N shares that pay dividends $\frac{1}{N}(Y_t - rB)$ in each period (or 0 if $rB \geq Y_t$). Suppose that the equilibrium prices for the two assets are $p_t^B(N, B, r, s_t)$ and $p_t^N(N, B, r, s_t)$. Then, the value of the tree for a particular structure is:

$$\Gamma = Bp_t^B(N, B, r, s_t) + Np_t^N(N, B, r, s_t)$$

We wish to maximize the value with respect to N, B, r, assuming that markets are complete with Arrow-Debreu trading.

Theorem 3 Modigliani-Miller. Γ *depends only on* s_t, *not on* B, N, r.

Proof. Using a no arbitrage argument, we are able to price the assets according to their payment streams, using equilibrium pricing kernels:

$$
\begin{aligned}
p_t^B &= \sum_{j=1}^{\infty} \int r Q_j(s_{t+j}, s_t) ds_{t+j} \\
p_t^N &= \sum_{j=1}^{\infty} \int \left(\frac{Y_{t+j} - rB}{N} \right) Q_j(s_{t+j}, s_t) ds_{t+j}
\end{aligned}
$$

Then, the value of the firm is:

$$
\begin{aligned}
Bp_t^B + Np_t^N &= \sum_{j=1}^{\infty} \int \left(Br + N \left(\frac{Y_{t+j} - rB}{N} \right) \right) Q_j(s_{t+j}, s_t) ds_{t+j} \\
&= \sum_{j=1}^{\infty} \int Y_{t+j} Q_j(s_{t+j}, s_t) ds_{t+j}
\end{aligned}
$$

which does not depend on B, N, r. ▮■

Note that this is a very simple model of an economy; this theorem need not hold in other models. Also, if it is certain that $Y_t \geq rB$, then the bond price depends only on r, while the stock price depends on all three other variables.

In the IID case with log utility,

$$
\begin{aligned}
p_t^B &= \sum_{j=1}^{\infty} E_t \left(r\beta^j \frac{u'(y_{t+j})}{u'(y_t)} \right) \\
&= \frac{rE(u'(y))}{u'(y_t)} \sum_{j=1}^{\infty} \beta^j \\
&= ry_t E\left(\frac{1}{y}\right) \frac{\beta}{1-\beta} \\
&= rp(y_t) E\left(\frac{1}{y}\right) \\
p_t^N &= \sum_{j=1}^{\infty} E_t \left(\frac{y_{t+j} - rB}{N} \beta^j \frac{y_t}{y_{t+j}} \right) \\
&= \sum_{j=1}^{\infty} E_t \left(\beta^j y_t \frac{1 - rB(1/y_{t+j})}{N} \right) \\
&= p(y_t)(1 - rBE(1/y)) \frac{1}{N}
\end{aligned}
$$

In general, holding an asset leads to:

- Capital gains or losses: These are measured by $\frac{p_{t+1}}{p_t}$.
- Interest or dividends: These are measured by $\frac{d_{t+1}}{p_t}$.

In this model, shares, trees, and bonds all have the same capital gains. Only some have risk, and the risk (and therefore the risk premium) varies. For bonds, the total return is $\frac{p^B_{t+1}}{p^B_t} + \frac{r}{r^B_t}$. As the amount paid out in bonds (called the *leverage*) increases, the shares become more risky, and therefore must have a better return.

2 Stochastic Growth Model

2.1 Time 0 Trading

In the stochastic growth model, we have:

- A discrete, finite state space.
- A representative household with preferences ordered by $\sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \pi_t(s^t) u(c_t(s^t), l_t(s^t))$; notice that decisions may depend on the entire history.
- Technology defined by $c_t(s^t) + x_t(s^t) \leq A_t(s^t) F(k_t(s^{t-1}), n_t(s^t))$ and $k_{t+1}(s^t) = (1 - \delta) k_t(s^{t-1}) + x_t(s^t)$, where
  - $x_t(s^t)$ is investment,
  - $A_t(s^t)$ is the exogenous productivity level, and
  - $k_t(s^{t-1})$ is this period's capital stock, which was decided in the previous period
- Endowments of k_0 and of $n_t(s^t) + l_t(s^t) = 1$ in each period.
- *Production firms* which hire $n_t(s^t), k^P_t(s^t)$ in order to produce and sell $c_t(s^t), x_t(s^t)$.
- *Investment firms* which own the capital and rent it to the production firms.
- Markets for:
  - Labor services: The household sells and the production firms buy at price $w^0_t(s^t)$
  - Capital services: Production firms buy and investment firms sell at price $r^0_t(s^t)$
  - Goods (for either consumption or investment): Households buy, production firms sell, and investment firms either buy or sell (depending on whether investment is positive or negative) at price $q^0_t(s^t)$.
  - Initial capital: Households sell, investment firms buy at price p_{k0}

- Firm Ownership: This turns out not to matter.

- Time 0, state-contingent trading.

Step 1: Solving the social planner's problem.

$$
\begin{aligned}
L &= \sum_{t=1}^{\infty} \beta^t \sum_{s^t \in S^t} \pi_t(s^t)(u(c_t(s^t), 1-n_t(s^t)) + \mu_t(s^t)(A_t(s^t)F(k_t(s^{t-1}), n_t(s^t)) \\
&\quad +(1-\delta)k_t(s^{t-1}) - c_t(s^t) - k_{t+1}(s^t)))
\end{aligned}
$$

(Because they are multiplied by $\beta^t \pi_t(s^t)$, the $\mu_t(s^t)$ are in terms of time t and state s^t prices instead of time 0 prices.) The optimal allocation must solve:

$$
\begin{aligned}
\frac{u_l(c_t(s^t), 1-n_t(s^t))}{u_c(c_t(s^t), 1-n_t(s^t))} &= F_n(k_t(s^{t-1}), n_t(s^t))A_t(s^t) \\
u_c(c_t(s^t), 1-n_t(s^t))\pi_t(s^t) &= \beta \sum_{s^{t+1}|s^t} u_c(c_{t+1}(s^{t+1}), 1-n_{t+1}(s^{t+1}))\pi(s^{t+1}) \\
&\quad \cdot (F_k(k_{t+1}(s^t), n_{t+1}(s^{t+1}))A_{t+1}(s^{t+1}) + 1 - \delta)
\end{aligned}
$$

where the first equations described the labor-leisure trade-off and the second equation described the intertemporal trade-off.

Step 2: Solving the optimization problems of households and the two types of firms. (Note that there are no risks in any of these equations because all the decisions are state-contingent at time 0.)

Households maximize $\sum_{t=1}^{\infty} \beta^t \sum_{s^t \in S^t} \pi_t(s^t) u(c_t(s^t), 1-n_t(s^t))$ subject to $\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} w_t^0(s^t)n_t(s^t) + k_0 p_{k_0}$. This leads to the first order conditions (if the solution is in the interior):

$$
\begin{aligned}
\beta^t \pi_t(s^t) u_c(c_t(s^t), 1-n_t(s^t)) &= \eta q_t^0(s^t) \\
-\beta^t \pi_t(s^t) u_l(c_t(s^t), 1-n_t(s^t)) &= \eta w_t^0(s^t)
\end{aligned}
$$

Production firms maximize $\sum_{t=0}^{\infty} \sum_{s^t} (q_t^0(s^t)(c_t(s^t)x_t(s^t)) - r_t^0(s^t)k_t^P(s^t) - w_t^0(s^t)n_t^0(s^t))$ subject to $c_t(s^t) + x_t(s^t) \leq A_t(s^t)F(k_t^P(s^t), n_t(s^t))$. Under constant returns to scale, $F(k,n) = nf(\frac{k}{n}) = nf(\hat{k})$. Then, the firms maximize $\sum_{t=0}^{\infty} \sum_{s^t} n_t(s^t)(q_t^0(s^t)A_t(s^t)f(\hat{k}^P(s^t)) - r_t^0(s^t)\hat{k}^P(s^t) - w_t^0(s^t))$, and the constraint is no longer needed. The optimal capital-labor ratio solves: $f'(\hat{k}^P(s^t)) = \frac{r_t^0(s^t)}{q_t^0(s^t)A_t(s^t)}$. Given prices, this completely determines the coefficient on $n_t(s^t)$. If the coefficient is positive, the firm would try to set n_t to be infinite. If the coefficient is negative, the firm would not produce. Thus, to ensure equilibrium with positive production, we must have the FOC's:

$$
\begin{aligned}
w_t^0(s^t) &= q_t^0(s^t)A_t(s^t)F_n(k_t(s^t), n_t(s^t)) \\
r_t^0(s^t) &= q_t^0(s^t)A_t(s^t)F_k(k_t(s^t), n_t(s^t))
\end{aligned}
$$

This does not determine n_t (but the rest of the economy will). Also, since there is free entry into the market, the maximum must be at 0.

Investment firms maximize $p_{k_0} k_o^I + \sum_{t=0}^{\infty} \sum_{s^t \in S^t} (r_t^0(s^t) k_t^I(s^{t-1}) - q_t^0(s^t)(k_{t+1}^I(s^t) - (1-\delta) k_t^I(s^{t-1})))$ by choosing $k_0^I, k_{t+1}^I(s^t)$. We rewrite this to consolidate the k_t:

$$(-p_{k_0} + r_0^0(s^0) + (1-\delta) q_0^0(s^0)) k_0^I + \sum_{t=0}^{\infty} \left(\sum_{s^t \in S^t} (r_t^0(s^t) + (1-\delta) q_t^0(s^t) k_t^I(s^{t-1})) - \sum_{s^{t-1} \in S^{t-1}} q_{t-1}^0(s^{t-1}) k_t^I(s^{t-1}) \right)$$

Consolidating shows that the coefficient on $k_t^I(s^{t-1})$ is $(\sum_{s^t|s^{t-1}} r_t^0(s^t) + q_t^0(s^t)(1-\delta)) - q_{t-1}^0(s^{t-1})$, which must be zero to ensure that the economy is in equilibrium. This yields the first order conditions:

$$\begin{aligned} p_{k_0} &= r_0^0(s^0) + (1-\delta) q_0^0 \\ q_{t-1}^0(s^{t-1}) &= \sum_{s^t|s^{t-1}} r_t^0(s^t) + q_t^0(s^t)(1-\delta) \end{aligned}$$

Step 3: We substitute the optimal allocation into the first order conditions to compute the (relative) equilibrium prices:

$$\begin{aligned} q_0^0(s^0) &= 1 \\ q_t^0(s^t) &= \beta^t \pi_t(s^t) u_c(s^t) \\ w_t^0(s^t) &= \frac{u_l(s^t)}{u_c(s^t)} q_t^0(s^t) = \frac{u_l(s^t)}{u_c(s^0)} \beta^t \pi_t(s^t) \\ r_t^0(s^t) &= \beta^t \pi_t(s^t) u_c(s^t) A_t(s^t) F_k(k_t(s^t), n_t(s^t)) \\ p_{k_0} &= r_0^0(s^0) + (1-\delta) q_0^0 \end{aligned}$$

In this model, if we re-opened markets at time $t < \tau$, the new prices for time t goods, $q_\tau^t(s^\tau)$ would simply have a new numeraire:

$$\begin{aligned} q_\tau^t(s^\tau) &= \frac{q_\tau^0(s^t)}{q_t^0(s^t)} = \beta^{\tau-t} \frac{u_c(s^\tau)}{u_c(s^t)} \pi_\tau(s^\tau|s^t) \\ w_\tau^t(s^\tau) &= \frac{w_\tau^0(s^t)}{q_t^0(s^t)} \\ r_\tau^t(s^\tau) &= \frac{r_\tau^0(s^t)}{q_t^0(s^t)} \end{aligned}$$

The implied wealth of the household at time t is:

$$\Upsilon(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} (q_\tau^t(s^t) c_\tau(s^\tau) - n_\tau(s^\tau) w_\tau^t(s^\tau))$$

which is future contingent claims less future obligations. Note that $\sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} q_\tau^t(s^t)c_\tau(s^\tau) = (r_t^t(s^t)+1-\delta)k_t(s^{t-1})$, so that non-labor wealth depends on the current value of capital (even though the households don't own the capital in this model).

2.2 Stochastic Growth with Sequential Trading

Suppose there is a pricing kernel, $\tilde{Q}_t(s_{t+1}|s^t)$, together with wage and rental rates, $\tilde{w}_t(s^t), \tilde{r}_t(s^t)$. Households may choose $\tilde{a}$, which are one-period contingent claims. Then, the household budget constraint and first order conditions become:

$$\begin{aligned}
\tilde{c}_t(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}(s_{t+1}|s^t)\tilde{Q}(s_{t+1}|s^t) &\leq \tilde{w}_t(s^t)\tilde{n}_t(s^t) + \tilde{a}_t(s_t|s^{t-1}) \\
\tilde{Q}_t(s_{t+1}|s^t) &= \beta\frac{u_c(s^{t+1})}{u_c(s^t)}\pi_t(s^{t+1}|s^t) \\
\tilde{w}_t(s^t) &= \frac{u_l(s^t)}{u_c(s^t)}
\end{aligned}$$

The production firms now maximize $\tilde{c}_t(s^t) + \tilde{x}_t(s^t) - \tilde{r}_t(s^t)k_t^P(s^t) - \tilde{w}_t(s^t)\tilde{n}_t(s^t)$. For the production firms to be in equilibrium, we have the zero profit conditions:

$$\begin{aligned}
\tilde{r}_t(s^t) &= A_t(s^t)F_k(s^t) \\
\tilde{w}_t(s^t) &= A_t(s^t)F_n(s^t)
\end{aligned}$$

Investment firms (assuming free entry, except that if they enter they are committed for two periods) maximize $-\tilde{k}_{t+1}^I(s^t) + \sum_{s^{t+1}|s^t}(\tilde{r}_{t+1}(s^{t+1})+1-\delta)\tilde{k}_{t+1}^I(s^t)\tilde{Q}(s_{t+1}|s^t)$. This yields the zero profit condition:

$$1 = \sum_{s_{t+1}}(\tilde{r}_{t+1}(s^{t+1})+1-\delta)\tilde{Q}(s_{t+1}|s^t)$$

We conjecture that:

$$\begin{aligned}
\tilde{Q}_t(s_{t+1}|s^t) &= q_{t+1}^t(s^{t+1}) \\
\tilde{w}_t(s^t) &= w_t^t(s^t) \\
\tilde{r}_t(s^t) &= r_t^t(s^t) \\
\tilde{a}_{t+1}(s_{t+1}|s^t) &= (r_{t+1}^{t+1}(s^{t+1})+1-\delta)k_{t+1}(s^t) \\
\tilde{a}_0 &= k_0(r_0^0(s^0)+1-\delta)
\end{aligned}$$

and we could check this by verifying that they lead to the same values of the choice variables.

In this model, trading in labor and capital happens during the period, so their trades are not state-contingent. However, assets are state-contingent.

We may also form the problem recursively, if we assume that:

- s^t follows a Markov process with transition probability $\pi(s_{t+1}|s_t)$.
- $A_t(s^t) = A(A_{t-1}(s^{t-1}), s_t)$ is Markov. (That allows s_t to be a growth rate.)

Then, the current state of the economy is $X = (s, A, K)$, where s is the current state, A is last period's productivity, and K is the capital stock at the beginning of the period. The household's state vector is (X, a), where a is the wealth of the household at the beginning of the period (which does not matter to the economy if households are homogeneous).

The social planner's value function is:

$$v(s, A, K) = \max \left(u(C, 1-N) + \beta \sum_{s' \in S} \pi(s'|s) E(V(K', S', s')) \right)$$

where the maximization is subject to the laws of motion:

$$\begin{aligned} K' + C &\leq AsF(K, N) + (1-\delta)K \\ A' &= As \\ s' &\sim \pi(s'|s) \end{aligned}$$

This yields the optimal policy functions, defined by:

$$\begin{aligned} C &= \Omega^C(X) \\ N &= \Omega^N(X) \\ K &= \Omega^K(X) \end{aligned}$$

The only state variable that households can choose is a (though they also choose their individual consumption and labor supply). Let $\overline{a}$ be the vector of arrow securities for next period. The households have the value function:

$$J(a, X) = \max_{c, n, \overline{a}} \left(u(c, 1-n) + \beta \sum_{x'} J(\overline{a}(x'), x') \hat{H}(X'|X) \right)$$

subject to the constraints, $c + \sum_{X'} Q(X'|X)\overline{a}(X') \leq w(X)n + a$ and that $\overline{a}(X')$ is greater than some borrowing constraint. (In equilibrium, $\overline{a}$ will depend on the value of capital.) This yields the optimal policy functions for the household:

$$\begin{aligned} c &= \sigma^c(a, X) \\ n &= \sigma^n(a, X) \\ \overline{a}(X') &= \sigma^a(X', a, X) \end{aligned}$$

Production firms deal only in the spot market, and maximize $c+x-r(X)k-w(X)n$ subject to $c+x \leq AsF(k,n)$. As before, this leads to the zero profit conditions:

$$\begin{aligned} r(X) &= AsF_k(k,n) \\ w(X) &= AsF_n(k,n) \end{aligned}$$

Investment firms have stochastic future profits; however, pricing kernels completely remove the uncertainty, since there are complete markets. That is, they must maximize $k'(-1 + \sum_{X'} Q(X'|X)(r(X')+1-\delta))$. This yields the zero profit condition:

$$1 = \sum_{X'} Q(X'|X)(r(X')+1-\delta)$$

(This requires that the price today (which is normalized to 1) equals the payoff tomorrow, discounted back by the pricing kernel.)

In equilibrium, markets clear, which requires equilibrium in the credit market:

$$\overline{a}(X') = (r(X')+1-\delta)K'$$

or, equivalently in the goods market:

$$c + x = AsF(K,N)$$

Substituting from the credit market, we check that the household budget constraint holds:

$$\begin{aligned} c + \sum_{X'} Q(X'|X)(r(X')+1-\delta)K' &= w(X)n + (r(X)+1-\delta)K \\ c + K' &= w(X)n + (r(X)+1-\delta)K \end{aligned}$$

Solving for K', substituting in equilibrium conditions ($K=k, N=n, C=c$) and decision rules, we find that:

$$\begin{aligned} K' &= w(X)n + (r(X)+1-\delta)K - c \\ &= AsF_n(k,n)n + (AsF_k(k,n)+1-\delta)K - c \\ &= AsF(K,N) + (1-\delta)K - C \\ &= AsF(K,\sigma^n(a,X)) + (1-\delta)K - \sigma^C(a,X) \\ &= AsF(K,\sigma^n((r(X)+1-\delta)K,X)) + (1-\delta)K - \sigma^c((r(X)+1-\delta)K,X) \\ &= G(X) \end{aligned}$$

This yields a law of motion for K.

Definition 4 *A* recursive competitive equilibrium *consists of:*

- *given prices, r, w, Q and perceived laws of motion for the state variables, $\hat{H}(X'|X)$, the decision rules, $\sigma^c, \sigma^n, \sigma^a(X', a, X)$ and the value function, $J(a, X)$, solve the household's problem,*

- *both types of firms are maximizing profits, which implies that*

$$
\begin{aligned}
r(X) &= AsF_K(K,N) = AsF_k(K, \sigma^n((r(X)+1-\delta)K, X)) \\
w(X) &= AsF_N(K,N) = AsF_n(K, \sigma^n((r(X)+1-\delta)K, X)) \\
1 &= \sum_{X'} Q(X'|X)(r(X)+1-\delta)
\end{aligned}
$$

(Note that these expressions depend only on the current state variables.)

- *The laws of motion of capital, rental rate, consumption, and labor satisfy $G(X)$ above.*

- *People have rational expectations about the state: $\pi(X'|X) = \hat{\pi}(X'|X)$.*

3 Government Finance

3.1 Lump Sum Taxation

For these models, we assume that the government spends $\{g_t\}$ in each period, which yields no utility for consumers. This is an exogenous stochastic process with $g_t < y_t$ for all t. The resource constraint is $y_t = c_t + g_t$. The only tax is a lump sum tax, τ_t. We assume that the government can borrow with 1-period state-contingent bonds, $b_t(s_{t+1})$. The bond pays off one unit of the consumption good next period if s_{t+1} is realized. In this model, the state is described by $s_t = (y_t, g_t)$, which we assume is Markov. The government budget constraint in state s_t is:

$$
g_t + b_{t-1}(s_t) = \tau_t + \int Q(s_{t+1}|s_t) b(s_{t+1}) ds_{t+1}
$$

where Q_1 is the one period ahead pricing kernel.

To find a competitive equilibrium in this economy, we recall that lump-sum taxes are non-distortionary (since they don't affect the Euler equations), so we apply the welfare theorems and solve the social planner's problem of maximizing $\sum_{t=0}^{\infty} E_0(\beta^t u(c_t))$, subject to $c_t \leq y_t - g_t$. By non-satiability, $c_t = y_t - g_t$. We then compute the representative agent's Euler equations and evaluate them at this allocation to compute prices and the equilibrium interest rate:

$$
\begin{aligned}
p_t &= E_t \left(\sum_{j=1}^{\infty} \beta^j \frac{u'(y_{t+j} - g_{t+j})}{u'(y_t - g_t)} y_{t+j} \right) \\
\frac{1}{R_{jt}} &= \beta^j E_t \left(\frac{u'(y_{t+j} - g_{t+j})}{u'(y_t - g_t)} \right)
\end{aligned}
$$

The level of government debt does not affect the interest rate (this is part of Ricardian equivalence). The pricing kernel is:

$$Q_j(s_{t+j}|s_t) = \beta^j \frac{u'(y_{t+j} - g_{t+j})}{u'(y_t - g_t)} f^j(s_{t+j}, s_t)$$

If we have time 0 trading, then the prices are:

$$\begin{aligned} q^0_{t+j}(s^{t+j}) &= Q_1(s_{t+j}|s_{t+j-1})...Q_1(s_1|s_0) \\ &= \beta^j \frac{u'(y_{t+j} - g_{t+j})}{u'(y_0 - g_0)} \end{aligned}$$

which is the price of one good in period $t+j$ if history $s^{t+j} = (s_0, ..., s_t, ...s_{t+j-1}, s_{t+j})$ occurs. Note that f^j is not used, since the entire history is specified in time 0 trading.

Government bonds are specified by:

$$b_{t-1}(s_t|s^{t-1}) = \tau_t(s^t) - g_t(s^t) + \int Q_1(s_{t+1}|s_t) b_t(s_{t+1}|s^t) ds_{t+1}$$

where taxes and bond holdings may depend on the entire history but government spending depends only on s_t and the price kernel depends only on s_t, s_{t+1} because the process is Markov.

To measure fiscal sustainability, we must discount the budget constraint at time $t+1$ by $Q_1(s_{t+1}|s_t)$; since the future state is unknown, we integrate over all possible s_{t+1}. This becomes:

$$\int Q_1(s_{t+1}|s_t) b_t(s_{t+1}|s^t) ds_{t+1} = \int Q_1(s_{t+1}|s_t)(\tau_{t+1} - g_{t+1}) ds_{t+1} - \int \int Q_1(s_{t+2}|s_{t+1}) b_{t+1}(s_{t+2}|s^{t+1}) ds_{t+2} ds_{t+1}$$

After continued substitution, this leads to the budget constraint:

$$\begin{aligned} b_{t-1}(s_t|s^{t-1}) &= \tau_t - g_t + \sum_{j=1}^{\infty} q^t_{t+j} \tau_{t+j}(s^{t+j}) d(s^{t+j} s^t) + \sum_{j=1}^{\infty} \int Q_j(s_{t+j}|s_t) g(s_{t+j}) ds_{t+j} \\ &\quad + \lim_{k\to\infty} \int q^t_{t+k+1}(s_{t+k+1}|s^{t+k}) b_{t+k}(s_{t+k+1}|s^{t+k}) d(s^{t+k+1}|s^t) \end{aligned}$$

where we define $\int x(s^{t+j} d(s^{t+j}|s^t) = \int \int ... \int x(s^{t+j}) ds_{t+j} ... ds_{t+2} ds_{t+1}$, which integrates over all possible futures that start with s^t. Notice that τ_t depends on the entire history, so that the associated prices and integrals must depend on the entire history as well, but g_t is Markov, so we can use a normal integral with a j-step pricing kernel.

In equilibrium, the limit term must be non-positive, since, otherwise, individuals would be accumulating assets (which they would rather consume). If the limit is negative, then the government is accumulating assets. This is not ruled out by equilibrium conditions, but we generally assume that it does not happen.

The household budget constraint is:

$$c_t(s^t)+\tau_t(s^t)+\sum_{j=1}^{\infty}\int(c_{t+j}(s^{t+j})+\tau_{t+j}(s^{t+j}))q^t_{t+j}(s^{t+j})d(s^{t+j}|s^t) \leq (p(s_t)+y(s_t))N_{t-1}(s^{t-1})+b_{t-1}(s_t|s^{t-1})+limit$$

Notice that intermediate bonds vanish in the present value budget constraint; initial trees and initial bonds describe a consumer's initial wealth. The limiting term must be non-positive by the transversality conditions and non-negative if we do not allow Ponzi schemes.

Proposition 5 Ricardian equivalence. *If the government satisfies its budget constraint, consumers' utility will not be affected by how the government finances itself through borrowing and taxes. That is, equilibrium prices and consumption depend only on* $\{y_t, g_t\}$

Proof. In equilibrium,

$$\begin{aligned} N_{t-1}(s^{t-1}) &= 1 \\ p(s_t) &= E_t(\sum_{j=1}^{\infty}\beta^j \frac{u'(y_{t+j}-g_{t+j})}{u'(y_t-g_t)}y_{t+j} \\ b_{t-1}d(s_t|s^{t-1}) &= b_{t-1}(s_t|s^{t-1}) \end{aligned}$$

We may replace $b_{t-1}(s_t|s^{t-1})$ from the government budget constraint to find that the household budget constraint is:

$$c_t(s^t)+\sum_{j=1}^{\infty}\int c_{t+j}(s^{t+j})q^t_{t+j}d(s^{t+j}|s^t) \leq y(s_t)-g(s_t)+\sum_{j=1}^{\infty}\int(y(s_{t+j})-g(s_{t+j}))Q_j(s_{t+j}|s_t)ds_{t+j}$$

which does not depend on taxes or bonds. ■ ■

One could try to measure fiscal sustainability by checking if debt is financed by future surpluses:

$$b_{t-1}(s_t) = \tau_t - g_t + \sum_{j=1}^{\infty}\frac{1}{R_{jt}}E(\tau_t - g_t)$$

where R_{jt} is the j-step ahead, risk-free interest rate. In a Lucas tree economy, the requirement above is not right, because covariance terms should be included, or there should be integrals over pricing kernels.

For a policy to be sustainable, the bonds must satisfy this equilibrium expression:

$$\begin{aligned} b_{t-1}(s_t|s^{t-1}) &= \tau_t(s^t) - g_t(s^t) + \sum_{j=1}^{\infty}E_t\left(\beta^j\frac{u'(y_{t+j}-g_{t+j})}{u'(y_t-g_t)}(\tau_{t+j}-g_{t+j})\right) \\ &= \tau_t(s^t) - g_t(s^t) + \sum_{j=1}^{\infty}\left(\frac{1}{R_{jt}}E_t(\tau_{t+j}-g_{t+j}) + Cov\left(\beta^j\frac{u'(y_{t+j}-g_{t+j})}{u'(y_t-g_t)}, \tau_{t+j}-g_{t+j}\right)\right) \end{aligned}$$

Since τ_{t+j} is not uniquely determined, the covariance term can be manipulated to be non-zero.

If there are non-altruistic generations, then we instead of *restricted Ricardian equivalence*, in which people are only indifferent between policies that have the same net present value during their lifetimes.

3.1.1 Taxes and Stochastic Growth

(Based on Bohn, 1995)

Suppose preferences are given by $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, and that the endowment obeys $\frac{y_t}{y_{t-1}} = \tilde{y}_t$, and $\tilde{y}_t$ is independent and identically distributed. We assume that $\frac{g_t}{y_t} = 1 - c$ is constant.

We consider two tax policies. In the first, $\tau_0 > 0$ and $\tau_t = 0$ for all future t. In this case, τ_0 may be greater than the endowment, but the government can then lend back to the people, so this is feasible. In this case, τ_0 is the present value of all government expenditures.

In the second policy, the government holds a constant debt-to-GDP ratio: $b = \frac{b_t/R_t}{y_t}$. We assume that all debt is one-period and risk free (not state contingent). The required debt will then imply the taxes for each period. Note that any constant debt policy is sustainable; however, discounting by the risk-free interest rate won't work:

$$\begin{aligned}
\lim_{j\to\infty} \frac{1}{R_{j+1,t}} E_t(b_{t+j}) &= \lim_{j\to\infty} \frac{1}{R_{j+1,t}} E_t(by_{t+j}R_{1,t+j}) \\
&= \lim_{j\to\infty} \frac{1}{R_1^j} E_t\left(R_1 by_t \prod_{i=1}^{j} \tilde{y}_{t+i}\right) \\
&= by_t \lim_{j\to\infty} \left(\frac{E(\tilde{y})}{R_1}\right)^j
\end{aligned}$$

As long as $R_1 > E(\tilde{y})$, this limit will be 0 (otherwise, it will be finite or infinite, either of which is a Ponzi scheme).

In either case, we compute the interest rate:

$$\begin{aligned}
\frac{1}{R_{jt}} &= \beta^j E_t(\frac{u'(y_{t+j} - g_{t+j})}{u'(y_t - g_t)} \\
&= \beta^j E_t((\frac{cy_{t+j}}{cy_t})^{-\gamma}) \\
&= \beta^j E_t(\prod_{i=1}^{j} \tilde{y}_{t+i}^{-\gamma}) \\
&= \beta^j \prod_{i=1}^{j} E_t(\tilde{y}_{t+i}^{-\gamma}) \\
&= (\frac{1}{R_{1t}})^j
\end{aligned}$$

so that risk-free interest rates are deterministic.

A constant debt-to-GDP ratio leads to a positive covariance (that is, a pro-cyclical policy), since for a lower value of $\tilde{y}_t$, $\frac{u'(y_{t+1}-g_{t+1})}{u'(y_t-g_t)}$ is higher while the debt is lower (since GDP is also lower) and taxes must be higher.

To check if a policy is sustainable, we should NOT discount future cash flows using the interest rates (R_t), but rather by the pricing kernel, q^t_{t+j}. (If you use the interest rates, this policy may seem unsustainable.) Using the pricing kernel, we must check that $\lim_{j\to\infty} q^t_{t+j}(s^{t+j})b_{t+j}(s^{t+j+1}) = 0$. This pricing kernel depends on both R_t and the covariance term. With a countercyclical policy, the covariance term is negative, which means that taxes seem higher than they need to be.

Note that all of this depends on taxes being lump sum. If taxes are distortionary, all of this may change.

In this economy, the interest rate depends on all of the primitives:

- $E(\tilde{y})$ scales everything.
- If $\gamma = Var(\tilde{y}) = 0$, then $R_1 = \frac{1}{\beta}$.
- If $\gamma = 0$, then $R_1 = \frac{1}{\beta}$ even as the variance increases.
- If $Var(\tilde{y}) = 0$, then the growth rate is constant but people want to smooth consumption, which means that the interest rate must be higher in equilibrium (to encourage saving).
- If both are non-zero, things get more complicated.

3.2 Optimal Distortionary Taxation

All of these strategies assume government commitment.

Suppose $\{g_t\}_{t=0}^{\infty}$ is given. The government imposes capital and labor taxes and may choose the sequences $\{\tau_t^n, \tau_t^k\}_{t=0}^{\infty}$. Preferences are given by $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$. The technology is $c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta)k_t$, where F has constant returns to scale. Households own the capital. The endowment is k_0 and $l_t + n_t = 1$.

Households have the one-period budget constraint:

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_t} \leq (1-\tau_t^n)w_t n_t + (1-\tau_t^k)r_t k_t + (1-\delta)k_t + b_t$$

where b_t are one period bonds, bought in time $t-1$ that pay off in time t (since there is a representative agent, all of the borrowing is between the household and the government). This gives the first order conditions:

$$\begin{aligned} \frac{u_l(c_t, 1-n_t)}{u_c(c_t, 1-n_t)} &= (1-\tau_t^n)w_t \\ \frac{u_c(c_t, 1-n_t)}{\beta u_c(c_{t+1}, 1-n_{t+1})} &= (1-\tau_{t+1}^k)r_{t+1} + 1 - \delta = R_t \end{aligned}$$

Firms have profit:

$$F(k_t, n_t) - r_t k_t - w_t n_t = (F_k(k_t, n_t) - r_t)k_t + (F_n(k_t, n_t) - w_t)n_t$$

by the Euler equation for constant returns to scale. In equilibrium, to ensure zero profits and finite production, we must have $F_k(k_t, n_t) = r_t$ and $F_n(k_t, n_t) = w_t$.

The government budget constraint is

$$g_t + b_t = \tau_t^k r_t k_t + \tau_t^n w_t n_t + \frac{b_{t+1}}{R_t}$$

In a *competitive equilibrium with distortionary taxes*,

- households are maximizing their utility, taking prices and government decisions as given,
- firms are maximizing profits,
- the government is obeying its budget constraint, taking household actions and all prices as given, and
- there are no Ponzi schemes in borrowing.

Notice that this does not uniquely determine an allocation, because there are many possible taxation schemes, and their distortionary nature means that there is no longer Ricardian equivalence.

Definition 6 *The* Ramsey allocation *is the competitive equilibrium with distortionary taxes that maximizes consumer utility.*

In this context, the *Ramsey problem* is to maximize $\sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$, subject first to feasibility conditions (both budget and resource constraints):

$$\begin{aligned}
\tau_t^k r_t k_t + \tau_t^n w_t n_t + \frac{b_{t+1}}{R_t} - b_t - g_t &= 0 \\
F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1} &= 0 \\
l_n + n_t &= 1 \\
(1-\tau_t^n) w_t n_t + (1-\tau_t^k) r_t k_t + (1-\delta)k_t + b_t - c_t - k_{t+1} - \frac{b_{t+1}}{R_t} &= 0
\end{aligned}$$

Note that the last budget constraint is actually redundant; if we sum the household budget constraint with the government budget constraint (and use the firm's zero profit conditions), we recover the feasibility constraint. This means we can remove any one of them. (This would no longer be true if the budget constraints were in present value terms; then we would have to keep the one-period feasibility constraint but could remove either of the others.) We also replace $l_t = 1 - n_t$. Because the allocation must be *implementable*, it must also satisfy all the first order conditions:

$$\begin{aligned}
u_l(c_t, 1-n_t) - u_c(c_t, 1-n_t)(1-\tau_t^n) w_t &= 0 \\
u_c(c_t, 1-n_t) - \beta u_c(c_{t+1}, 1-n_{t+1}) \left((1-\tau_{t+1}^k) r_{t+1} + 1 - \delta \right) &= 0 \\
R_t - \left((1-\tau_{t+1}^k) r_{t+1} + 1 - \delta \right) &= 0 \\
F_k(k_t, n_t) - r_t &= 0 \\
F_n(k_t, n_t) - w_t &= 0
\end{aligned}$$

Note that the government's only choice variables are τ_t^k, τ_t^n, but we maximize over c_t, n_t, k_{t+1}, r_t, w_t, τ_t^k, τ_t^n, b_{t+1}, R_t, and l_t (many of which are implied once others are known).

In an open economy, interest rates and rental rates are exogenously determined by the world economy. Furthermore, the household and government budget constraints no longer imply the usual resource constraint, since government and household bond holdings do not have to sum to 0. Furthermore, household and firm capital now satisfy:

$$c_t + k_{t+1} + g_t + (k_{t+1}^H - k_{t+1}^F) + (b_t^G - b_t^H) = F(t) + \frac{b_{t+1}^G - b_{t+1}^H}{R_t}(1 - \delta + r_t^*)(k_t^H - k_t) + (1-\delta)k_t$$

3.2.1 Chamley Approach

Define the after-tax wage and rental rates by:

$$\begin{aligned}\tilde{r}_t &= (1-\tau_t^k)r_t \\ \tilde{w}_t &= (1-\tau_t^n)w_t\end{aligned}$$

Then, the Euler equations become:

$$\begin{aligned}u_l(c_t,1-n_t)-u_c(c_t,1-n_t)\tilde{w}_t &= 0 \\ u_c(c_t,1-n_t)-\beta u_c(c_{t+1},1-n_{t+1})(\tilde{r}_{t+1}+1-\delta) &= 0 \\ R_t-(\tilde{r}_{t+1}+1-\delta) &= 0\end{aligned}$$

and the tax revenues and government budget constraint become:

$$\begin{aligned}\tau_t^k r_t k_t+\tau_t^n w_t n_t=(r_t-\tilde{r}_t)k_t+(w_t-\tilde{w})n_t &= F(k_t,n_t)-\tilde{r}_t k_t-\tilde{w}_t n_t \\ F(k_t,n_t)-\tilde{r}_t k_t-\tilde{w}_t n_t+\frac{b_{t+1}}{R_t}-b_t-g_t &= 0\end{aligned}$$

Substituting in for r_t, w_t (using the zero profit conditions) and for the taxes leaves the following constraints:

$$\begin{aligned}F(k_t,n_t)-\tilde{r}_t k_t-\tilde{w}_t n_t+\frac{b_{t+1}}{R_t}-b_t-g_t &= 0 \\ F(k_t,1-n_t)+(1-\delta)k_t-c_t-g_t-k_{t+1} &= 0 \\ u_l(c_t,1-n_t)-u_c(c_t,1-n_t)(1-\tau_t^n)w_t &= 0 \\ u_l(c_t,1-n_t)-u_c(c_t,1-n_t)\tilde{w}_t &= 0 \\ u_c(c_t,1-n_t)-\beta u_c(c_{t+1},1-n_{t+1})(\tilde{r}_{t+1}+1-\delta) &= 0 \\ R_t-(\tilde{r}_{t+1}+1-\delta) &= 0\end{aligned}$$

We then maximize the Lagrangian:

$$\begin{aligned}L \;=\; & \sum_{t=0}^{\infty}\beta^t(u(t)+\psi_t\left(F(t)-\tilde{r}_t k_t-\tilde{w}_t n_t+\frac{b_{t+1}}{R_t}-b_t-g_t\right)+\theta_t\left(F(t)+(1-\delta)k_t-c_t-g_t-k_{t+1}\right) \\ & +\mu_{1t}\left(u_l(t)-u_c(t)\tilde{w}_t\right)+\mu_{2t}\left(u_c(t)-\beta u_c(t+1)(\tilde{r}_t+1-\delta)\right)+\mu_{3t}\left(R_t-(\tilde{r}_{t+1}+1-\delta)\right))\end{aligned}$$

The first order condition with respect to capital is:

$$-\beta^t\theta_t+\beta^{t+1}(\psi_{t+1}(F_k(t+1)-\tilde{r}_t)+\theta_{t+1}(F_k(t+1)+1-\delta))=0$$

With no taxation, this would simply be:

$$-\beta^t\theta_t+\beta^{t+1}\theta_{t+1}(F_k(t+1)+1-\delta)=0$$

The coefficient on the extra constraint measures the additional value of the taxes that the government collects off of extra capital.

Suppose g_t becomes constant for all $t > T$ and that a steady state exists. Then, we must have $\theta_t \to \theta$ and $\psi_t \to \psi$, which yields the first order condition:

$$\theta = \beta(\psi(r - \tilde{r}) + \theta(r + 1 - \delta))$$

In the steady state, the second Euler equation tells us that:

$$\begin{aligned} u_c &= \beta u_c(\tilde{r} + 1 - \delta) \\ \frac{1}{\beta} - \tilde{r} &= 1 - \delta\theta = \beta(\psi(r - \tilde{r}) + \theta(r - \tilde{r})) \\ 0 &= \beta(\psi + \theta)(r - \tilde{r}) \end{aligned}$$

By non-satiation, $\theta > 0$. Furthermore, $\beta > 0, \psi \geq 0$. This means that $r = \tilde{r}$ in the steady state, and the steady state capital tax must be 0 for the optimal allocation.

Zero capital taxation means that intertemporal substitution is not changed in the steady state. It is also related to the persistence of capital (though it would still hold if $\delta = 1$).

3.2.2 Primal Approach

To use the primal approach:

1. Obtain the first order conditions of the households and firms, and use these to solve for the prices and taxes as functions of the allocations.
2. Use these equations to eliminate prices and taxes from the household's present value budget constraint. This yields the *implementability constraint*.
3. Solve the Ramsey problem by maximizing utility subject to the resource and implementability constraints.
4. Use this allocation to back out prices and taxes.

(This always uses present value budget constraints, which is equivalent to the previous approach, as long as b_t is not restricted.) This imposes the first order conditions from the household and firm before optimization, which removes prices, turning the problem into a social planner problem, with the additional implementability condition.

The household's present value budget constraint and first order conditions are:

$$\begin{aligned}
\sum_{t=0}^{\infty} q_t^0 c_t &= \sum_{t=0}^{\infty} q_t^0((1-\tau_t^n)w_t n_t) + (1-\tau_0^k+1-\delta)k_0 + b_0 \\
q_t^0 &= \prod_{i=0}^{t} R_i^{-1}, q_0^0 = 1 \\
q_t^0 &= \beta^t \frac{u_c(t)}{u_c(0)} \\
(1-\tau_t^n)w_t &= \frac{u_l(t)}{u_c(t)} \\
q_t^0(1-\tau_t^n)w_t &= \beta^t \frac{u_l(t)}{u_c(0)}
\end{aligned}$$

All of the capital and bond terms after time 0 cancel by the no-arbitrage conditions. For example, combining the budget constraints across two periods:

$$\begin{aligned}
c_t + k_{t+1} + \frac{b_{t+1}}{R_t} &= w_t n_t + r_t k_t + (1-\delta)h_t + b_t \\
c_t + \frac{1}{R_t}c_{t+1} + \frac{b_{t+1}}{R_t} + \frac{b_{t+2}}{R_{t+1}} &= w_t n_t + \frac{w_{t+1}n_{t+1}}{R_t} + \left(\frac{r_t+1-\delta}{R_t} - 1\right)k_{t+1} + r_t k_t + (1-\delta)k_t + b_t
\end{aligned}$$

The coefficient on k_{t+1} must be 0 to keep the budget constraint bounded. Using similar logic leads to only k_0 appearing in the present value budget constraint.

As before, the firm has zero-profit conditions:

$$\begin{aligned}
r_t &= F_k(t) \\
w_t &= F_n(t)
\end{aligned}$$

These equations eliminate prices and taxes.

Since k_0 is given, τ_0^k acts like a lump-sum tax and is not distortionary. Thus, it would be optimal (but trivial) to choose $\tau_0^k k_0 = \sum_{t=0}^{\infty} q_t^0 g_t$. Instead, we require that $\tau_0^k \le \bar{\tau}_0^k$. τ_0^k will always be maximized in an optimal solution.

Substituting for prices and wages in the household budget constraint yields the implementability constraint:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \frac{u_c(t)}{u_0(t)} c_t &= \sum_{t=0}^{\infty} \frac{u_l(t)}{u_c(0)} \beta^t n_t + (1-\tau_0^k+1-\delta)k_0 + b_0 \\
\sum_{t=0}^{\infty} \beta^t (u_c(t)c_t - u_l(t)n_t) - A &= 0 \\
A &= u_c(0)(k_0(1-\tau_0^k+1-\delta) + b_0)
\end{aligned}$$

Then, the Lagrangian for the Ramsey problem is:

$$\begin{aligned}
L &= \sum_{t=0}^{\infty}\beta^t u(t) + \sum_{t=0}^{\infty}\beta^t\theta_t(F(t)-(1-\delta)k_t - c_t - g_t - k_{t+1}) + \phi\left(\sum_{t=0}^{\infty}\beta^t(u_c(t)c_t - u_l(t)n_t) - A\right) \\
&= \sum_{t=0}^{\infty}\beta^t\left(V(c_t,n_t,\phi) + \theta_t(F(t)+(1-\delta)k_t) - c_t - g_t - k_{t+1}\right) - \phi A \\
V(c_t,n_t,\phi) &= u(c_t,1-n_t) + \phi(u_c(t)c_t - u_l(t)n_t)
\end{aligned}$$

The first order conditions with respect to $c_t, n_t, k_{t+1}, c_0, n_0$ are:

$$\begin{aligned}
V_c(t) &= \theta_t \\
V_n(t) &= -\theta_t F_n(t) \\
\theta_t &= \beta\theta_{t+1}(F_k(t+1)+1-\delta) \\
V_c(0) &= \theta_0 + \phi A_c \\
V_n(0) &= -\theta_0 F_n(0) + \phi A_n
\end{aligned}$$

which shows that the optimal allocation is characterized by:

$$\begin{aligned}
V_c(t) &= \beta V_c(t+1)(F_k(t+1)+1-\delta) \\
V_n(t) &= -V_c(t)F_n(t) \\
V_n(0) &= (\phi A_c - V_c(0))F_n(0) + \phi A_n
\end{aligned}$$

We can back out prices and taxes from this.

If a steady state exists, the tax on capital satisfies:

$$\frac{1}{\beta} = (1-\tau^k)F_k + 1 - \delta$$

However, $\overline{V_c} = \beta\overline{V_c}(F_k+1-\delta)$, which means that we must have $\frac{1}{\beta} = F_k + 1 - \delta$, and the steady state capital must still be 0.

3.2.3 Consumption Taxes

If there is a consumption tax instead of a capital tax, the budget constraint and implementability conditions become:

$$\begin{aligned}
\sum_{t=0}^{\infty} q_t^0(1+\tau_t^0)c_t &= \sum_{t=0}^{\infty} q_t^0(1-\tau_t^n)n_t w_t + (r_0+1-\delta)k_0 + b_0 \\
\sum_{t=0}^{\infty}\beta^t(u_c(t)c_t - u_l(t)n_t) - A &= 0 \\
A &= u_c(0)\left((F_k(0)+1-\delta)k_0 + b_0\right)\frac{1}{1+\tau_0^c}
\end{aligned}$$

If $\tau_0^c = 0$, then this reduces to the problem with $\tau_t^k = 0$; more generally, for any τ_k^0, we can find τ_c^0 that will lead to the same A (the value will depend on k_0, b_0, δ).

Now, we write prices as:

$$
\begin{aligned}
q_t^0 &= \beta^t \frac{u_c(t)}{u_c(0)} \left(\frac{1+\tau_0^c}{1+\tau_t^c} \right) \\
R_t &= \frac{u_c(t)}{\beta u_c(t+1)} \\
\frac{1+\tau_{t+1}^c}{1+\tau_t^c} &= r_{t+1} + 1 - \delta \\
w_t &= \frac{u_l(t)}{u_c(t)} \frac{1+\tau_t^c}{1-\tau_n^t}
\end{aligned}
$$

Comparing steady states, we see that, for any steady state in c, n, g, k, the capital tax is given by $\frac{u_c}{\beta u_c} = (1-\tau_{t+1}^k)F_k + 1 - \delta$, which means that the capital tax is also constant. For a consumption tax, instead, we must have $\frac{u_c}{\beta u_c} \frac{1+\tau_{t+1}^c}{1+\tau_t^c} = F_k + 1 - \delta$, which means that the ratio of consumption taxes must be constant in the steady state, and $\Delta(1+\tau_t^c)$ is constant. If τ_k is positive in the steady state, this corresponds to $\Delta(1+\tau_t^c) > 0$, so that the tax rate would increase forever, which is not feasible. This is another reason that the steady state capital tax is 0. In the steady state, $w, \frac{u_l}{u_c}$ are also constant, so $1-\tau_t^n$ must also grow at a constant rate. This would lead to a labor subsidy to offset the growing consumption tax. In contrast, in the optimal steady state, τ_c is constant (but need not be 0).

(In fact, since there are two Euler equations for the households, and two of labor, capital, and consumption taxes can be used to achieve the same outcome.)

If human capital accrues the way capital does (that is, it disappears in present value), then the steady state labor tax is 0 as well, which means that the government would have to tax a lot initially.

3.2.4 Optimal Taxation Under Uncertainty

(This follows Lucas and Stokey.)

In this economy, the technology is $c_t(s^t) + g_t(s^t) = n_t(s^t)$, the endowment is $n_t(s^t) + l_t(s^t) = 1$. We assume that there are complete credit markets, so that the government may have state-contingent debt. (This gets tougher if all debt is risk-free.) Taxes are also state-contingent (and government spending depends on the state but is exogenous). This yields the government budget constraint:

$$g_t(s^t) + b_t(s_t|s^{t-1}) = \tau_t^n(s^t)w_t(s^t)n_t(s^t) + \sum_{s_{t+1}} b_{t+1}(s_{t+1}|s^t)p_t(s_{t+1}|s^t)$$

We set $q_{t+1}^0(s^{t+1}) = q_t^0(s^t)p_t(s_{t+1}|s^t)$.

Firms maximize $c_t(s^t) + g_t(s^t) - w_t(s^t)n_t(s^t)$ subject to $c_t(s^t) + g_t(s^t) \leq n_t(s^t)$. This yields the zero profit condition $w_t(s^t) = 1$.

The household present value budget constraint and first order conditions are:

$$\begin{aligned}
\sum_{t=0}^{\infty}\sum_{s^t\in S^t} q_t^0(s^t)c_t(s^t) &= b_0 + \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\left(1-\tau_t^n(s^t)\right)n_t(s^t)(1)q_t^0(s^t) \\
q_t^0(s^t) &= \beta^t\pi_t(s^t)\frac{u_c(s^t)}{u_0(s^0)} \\
\frac{u_l(s^t)}{u_c(s^t)} &= 1-\tau_t^n(s^t)
\end{aligned}$$

Combining all of these conditions yields the implementability condition:

$$\sum_{t=0}^{\infty}\beta^t\pi_t(s^t)(u_c(s^t)c_t(s^t) - u_l(s^t)n_t(s^t)) - b_0u_c(s^0) = 0$$

This gives the social planner's Lagrangian:

$$\begin{aligned}
L &= \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\left(u(s^t) + \phi\left(u_c(s^t)c_t(s^t) - u_l(s^t)n_t\right) + \theta_t(s^t)\left(n_t(s^t) - c_t(s^t) - g_t(s^t)\right)\right) \\
&\quad -\phi u_c(s^0)b_0
\end{aligned}$$

(The sign on the Lagrange multipliers for the household budget constraint suggests that the constraint is actually $Spending \geq Income$. Since the household will spend all its income anyway, this will make the multipliers non-negative for the social planner. The government budget constraint is of the form $Taxes - Speing$ as usual.) Then, the first order conditions with respect to $c_t(s^t), n_t(s^t), c_0(s^0), n_0(s^0)$ are:

$$\begin{aligned}
u_c(s^t) + \phi u_c(s^t) + \phi(u_{cc}(s^t)c_t(s^t) - u_{lc}(s^t)n_t(s^t)) - \theta_t(s^t) &= 0 \\
-u_l(s^t) - \phi u_l(s^t) + \phi(-u_{lc}(s^t)c_t(s^t) + u_{ll}(s^t)n_t(s^t)) + \theta_t(s^t) &= 0 \\
u_c(s^0) + \phi u_c(s^0) + \phi(u_{cc}(s^0)c_o(s^0) - u_{lc}(s^0)n_t(s^0)) - \theta_0(s^0) - \phi u_{cc}(s^0)b_0 &= 0 \\
-u_l(s^0) - \phi u_l(s^0) + \phi(-u_{lc}(s^0)c_t(s^0) + u_{ll}(s^0)n_t(s^0)) + \theta_0(s^0) + \phi u_{cl}(s^0)b_0 &= 0
\end{aligned}$$

which characterizes the optimal solution(s). Substituting for $\theta_t(s^t)$, we find, for all $t \geq 1$:

$$(1+\phi)u_c(s^t) + \phi(u_{cc}(s^t)c_t(s^t) - u_{lc}(s^t)n_t(s^t)) = (\phi+1)u_l(s^t) + \phi(u_{cl}(s^t)c_t(s^t) - u_{ll}(s^t)n_t(s^t))$$

The only unknowns in this equation are ϕ and $c_t(s^t)$, since we may write $n_t(s^t)$ in terms of $c_t(s^t)$ and the known $g_t(s^t)$.

The Laffer curve plots revenues versus tax rates. If the government spending lies completely above the Laffer curve, it is infeasible. In other cases, the government spending line intersects the curve in two places. If there are multiple solutions for the tax rate, then check them in the resource constraint. One might lead to negative consumption. (That solution might minimize consumer utility instead of maximizing it.) This curvature occurs because the problem is no longer concave, so that there can be more than one solution.

Suppose $g_t(s^t) = g_{t+k}(s^{t+k})$. Since ϕ is constant, the equation above implies that $c_t(s^t) = c_{t+k}(s^{t+k})$. Thus, whenever the government expenditure is the same, the allocation (and therefore the prices and taxes) should be the same as well. This means there is no history dependence in the allocation. (This depends on having complete markets.)

For example, if government spending is constant across time, then c_t, n_t, τ_t^n are constant across time. If $b_0 = 0$, then the tax rate must satisfy $\tau^n = \frac{g}{n}$.

If we substitute:

$$\begin{aligned} \frac{u_l(s^t)}{u_c(s^t)} &= 1 - \tau_t^n(s^t) \\ n_t(s^t) - g_t(s^t) &= c_t(s^t) \end{aligned}$$

into the implementability constraint, we find that:

$$\begin{aligned} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u_c(s^t) \left(n_t(s^t) - g_t(s^t) - \left(1 - \tau_t^n(s^t)\right) n_t(s^t) \right) + b_0 u_c(0) &= 0 \\ \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u_c(s^t) \left(\tau_t^n(s^t) n_t(s^t) - g_t(s^t) \right) - u_c(s^0) b_0 &= 0 \end{aligned}$$

(The inner term looks like the individual terms of the government budget constraint.) Multiplying the first order conditions by $c_t(s^t)$ and $n_t(s^t)$ respectively and summing, we find:

$$\begin{aligned} (1+\phi)(u_c(s^t)c_t(s^t) - u_l(s^t)n_t(s^t)) + \phi(c_t(s^t)^2 u_{cc}(s^t) - 2c_t(s^t)n_t(s^t)u_{lc}(s^t) + n_t(s^t)^2 u_{ll}(s^t)) & \\ -\theta_t(s^t)(c_t(s^t) - n_t(s^t)) &= 0 \end{aligned}$$

Since utility is strictly concave, $Q = c_t(s^t)^2 u_{cc}(s^t) - 2c_t(s^t)n_t(s^t)u_{lc}(s^t) + n_t(s^t)^2 u_{ll}(s^t) < 0$. Furthermore, $\theta_t(s^t) > 0$ (by non-satiation) and $\phi > 0$ (as long as the present value of government expenditures exceeds its initial assets, $-b_0$).

Suppose g_t is zero except in period T when it is $g_t = g > 0$; this is a perfectly foreseen expense. Again, when $t \neq T$, consumption, labor and taxes will all be constant, both before and after the expense. Furthermore, because of the sign restrictions in the equation above and because $c_t = n_t$ (since $g_t = 0$), we must have:

$$c_t(s^t)u_c(s^t) - n_t(s^t)u_l(s^t) > 0$$

for all $t \neq T$. This means that $\frac{u_l(s^t)}{u_c(s^t)} < 1$ and therefore $\tau_t^n(s^t) > 0$. This means that taxes are collected every period, which smooths the taxes needed for the later government spending. At time T, $c_t(s^t) < n_t(s^t)$, and $-\theta_t(s^t)(c_t(s^t) - n_t(s^t)) > 0$ while $\phi Q < 0$. This makes the sign of $c_t(s^t)u_c(s^t) - n_t(s^t)u_l(s^t)$ indeterminate. Depending on the level of government spending and the particular utility function, taxes at time T could be positive, negative, or 0. This shows that, in this scenario, the government taxes before it must spend, goes into debt for the spending, and then uses future taxes to pay the interest on the debt as the debt's present value goes to 0 (it never actually pays off the debt, because then it would be optimal to stop taxing).

Suppose $g_t = 0$ for all $t \neq T$ and $g_T > 0$ with probability α. As before, the allocation and taxes will be the same for all $t \neq T$, as well as in period T if $g_T = 0$. Again, the tax rate will be positive for those periods. In this case, the government taxes in each period and then buys Arrow securities at time $T-1$ that will pay off if $g_T > 0$ and will go further into debt if $g_T = 0$. In either case, the government will pay interest on the (same) debt with future taxes. Again, after T, the past doesn't matter, even without the process being Markov.

Complete markets allow the debt to depend on the outcomes of g_t. This is what allows the paths of taxes and consumption not to depend on history. If markets are not complete, debt may become history dependent.

Once we have found the optimal consumption, labor and taxation paths, we may read off the path of debt, $b_{t+1}(s_{t+1}|s^t)$, from $\{\tau_t n_t - g_t\}$ and the interest rates. We may compute debt backward, based on $b_0, \tau_1, g_1, \tau_2, g_2, \ldots$, or forward, with the equation:

$$\begin{aligned} b_t(s_t|s^{t-1}) &= \tau_t^n(s^t)n_t(s^t) - g_t(s^t) + \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} (\tau_{t+j}^n(s^{t+j})n_{t+j}(s^{t+j}) - g_{t+j}(s^{t+j}))q_{t+j}^t(s^{t+j}) \\ &= \sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_{t+j}(s^{t+j}|s^t) \frac{u_c(s^{t+j})}{u_c(s^t)} \left(\left(1 - \frac{u_l(s^{t+j})}{u_c(s^{t+j})}\right) \left(c_{t+j}(s^{t+j}) + g_{t+j}(s^{t+j})\right) - g_{t+j}(s^{t+j}) \right) \end{aligned}$$

where the second equality follows by substituting in the equilibrium expressions for prices, taxes, and labor. From this expression, we see that $\{g_{t+j}\}$ determines the future path of consumption and therefore the value of future debt. Furthermore, $g_t(s^t)$ depends only on the current state, so that if s_t is Markov, then debt must be Markov as well. In this case, $b(s_t|s^{t-1})$ depends only on s_t, and there is no history dependence.

In a stochastic world, there is no steady state. Instead we may do asymptotics (Zhu, 1992). Given the household first order condition:

$$u_c(s^t) = \beta E_t(u_c(s^{t+1})((1 - \tau_{t+1}^k(s^{t+1}))r_t(s^t) + 1 - \delta)$$

and the Ramsey allocation, $\{c_t(s^t), n_t(s^t), k_{t+1}(s^t)\}_{s^t,t}$, we may derive pricing kernels, $p(s_{t+1}|s^t)$, wages, rents, taxes, and bond holdings. However, there is indeterminacy with respect to capital taxes, $\tau_t^k(s^t)$ and bond holdings, $b_{t+1}(s_{t+1}|s^t)$; given any solution, we may find an alternative. Let $\{\epsilon_t(s^t)\}$ be any stochastic process with $E_t(u_c(s^{t+1})\epsilon_t(s^t)r_{t+1}(s^{t+1})) = 0$. Define:

$$\begin{aligned}
\hat{\tau}_0^k &= \tau_0^k \\
\hat{\tau}_{t+1}^k(s^{t+1}) &= \tau_{t+1}^k(s^{t+1}) + \epsilon_{t+1}(s^{t+1}) \\
\hat{b}_{t+1}(s_{t+1}|s^t) &= b_{t+1}(s_{t+1}|s^t) + \epsilon_{t+1}(s^{t+1})r_{t+1}(s^{t+1})k_{t+1}(s^t)
\end{aligned}$$

For this new allocation:

- Government capital tax revenue and maturing government debt are unchanged.
- Government debt carried from time t to $t+1$ have the same present value at time t, evaluated at the original prices.
- The household intertemporal Euler equation continues to hold.

4 Labor Markets: the Lucas-Prescott Island Model

Suppose we have:

- a continuum of agents,
- a large number of spatially separated islands, each with technology $\theta f(n)$, where f is constant across islands and satisfies the Inada conditions,
- $\theta_1 < ... < \theta_m$, with $\pi(\theta'|\theta)$ with $P(\theta' \leq \theta_k|\theta) = \sum_{i=1}^{k} \pi(\theta, \theta_i)$ a decreasing function of θ (so that good and bad times tend to be persistent),
- risk-neutral preferences, $\sum_{t=0}^{\infty} \beta^t c_t$, and
- a moving technology so that, if a worker wants to move, it takes one period (of foregone wages).

Let βV_u be the value of moving (it is discounted by β because of the lost wages during moving). In equilibrium, this will be equal across all individuals (since everyone has the same information and therefore would pick the same optimal islands(s)). βV_u includes all the information about the distribution of populations and productivity across islands.

Notice that after a productivity change, it takes one period for the labor force to change. (It is also possible that productivity changes in two consecutive periods, so that people move in and then out.)

Let x be the population on the island at the beginning of a period and θ the current productivity level on the island. (Though we assume all workers know the productivity levels on all the islands, only one's own island appears in the value function.) Let $w(\theta, x) = \theta f'(n(\theta, x))$ be the wage on the island; this depends on productivity and on the number of employed workers, $n(\theta, x)$ (note that $n(x, \theta) \leq x$, since the number of workers can't exceed the labor force at the beginning of the period). Assuming that V_u is known, the value function is:

$$V(\theta, x) = \max\{\beta V_u, w(\theta, x) + \beta E(V(\theta', x')|\theta, x)\}$$

The first option corresponds to moving, while the second option corresponds to working this period and staying on the island. We consider 3 cases:

- $n < x$**:** In this case, people are leaving the island, so $V(\theta, x) = \beta V_u$. However, not everyone leaves (because of the Inada conditions). Thus, we must have $w(\theta, x) + \beta E(V(\theta', x')|\theta, x) = \beta V_u$.
- $n = x$**:** Since everyone is working, we know that the wage must be $w(\theta, x) = \theta f'(x)$.
  - $x' > x$**:** Since people are moving in, we must have $E(V(\theta', x')|\theta, x) = \beta V_u$. This means that the value function is $V(x, \theta) = \theta f'(x) + \beta V_u$.
  - $x = x'$**:** This determines the population next period, so that $V(x, \theta) = \theta f'(x) + \beta E(V(\theta', x)|\theta)$.

This yields the value function:

$$V(\theta, x) = \max\{\beta v_u, \theta f'(x) + \beta \min\{V_u, E(V(\theta', x)|\theta)\}\}$$

We then use the three cases to determine the equilibrium flows between islands:

- $n < x$**:** In this case, $x' = n$. We choose n so that people are indifferent between moving and staying:
  $$\beta V_u = \theta f'(n) + \beta E(V(\theta', n)|\theta)$$
  We define $x^+(\theta)$ as the n that solves this equation for a given θ.
- $n = x, x' > x$**:** Note that, no matter the current value of x, x' will be the same; otherwise, more people would move in. This means that $\beta V_u = \beta E(V(\theta', x')|\theta)$. This equation determines $x^-(\theta) = x'$.
- $n = x, x' = x$**:** In this case, there is no movement. This will only happen if $x^-(\theta) \leq x \leq x^+(\theta)$. This defines a *range of inactivity* in which wages are high enough that no one moves out but not so high that people move in.

Holding V_u fixed, $x^-(\theta) < x^+(\theta)$ and both curves are upward-sloping in θ (since more productivity will lead to a better wage with the same number of workers). For any x, the population will move to the closer end of the range of inactivity.

If $x^+(\theta_1) \geq x^-(\theta_m)$, then any $x \in [x^-(\theta_m), x^+(\theta_1)]$ is in the range of inactivity. Thus, once the population reaches this interval, it will never move again. This is an equilibrium with no labor movement. On the other hand, if $x^+(\theta_1) \leq x^-(\theta_m)$, then the ergodic set will be $x \in \{x^-(\theta_i), x^+(\theta_i)\}$, with $x^+(\theta_1) \leq x \leq x^+(\theta_m)$ (since once x is inside this range, it will never go below $x^+(\theta_1)$). Since V_u is fixed in equilibrium, this is a finite number of possible points. Note that not all transitions of θ will lead of a movement, if $[x^-(\theta_i), x^+(\theta_i)]$ and $[x^-(\theta_j), x^+(\theta_j)]$ overlap. In either case, the stationary equilibrium is an equilibrium across all of the islands; an individual island's productivity and therefore population keep changing, but the distribution of productivity and population is constant across all islands.

βV_u determines the welfare of people on the islands the the population and on the non-optimal islands (?). If βV_u increases, the population on non-optimal islands increases.

Given V_u, we may compute the transition function:

$$\begin{aligned}\Gamma(\theta', x'|\theta, x) &= \pi(\theta'|\theta)I(x' = x^+(\theta), x > x^+(\theta)) \\ &\quad +\pi(\theta'|\theta)I(x' = x^-(\theta), x < x^-(\theta)) \\ &\quad +\pi(\theta'|\theta)I(x' = x, x^-(\theta) \leq x \leq x^+(\theta))\end{aligned}$$

Then,the distribution of population across islands can be found from the transition function:

$$\Psi_{t+1}(\theta', x'; V_u) = \sum_\theta \sum_x \Gamma(\theta', x'|\theta, x)\Psi_t(\theta, x; V_u)$$

We iterate on Ψ_t until we find the stationary distribution, $\Psi(x, \theta; V_u)$. Then, the population must be

$$\overline{X} = \sum_{x \in X} \sum_{\theta_1, \ldots, \theta_m} x\Psi(\theta, x; V_u)$$

Each value of V_u can be mapped to some $\overline{x}$. Thus, given the actual population size, we may determine V_u in equilibrium.

(Remember that if Γ is a transition function, the stationary distribution satisfies $q'\Gamma = q'$.)

Because we have made minimal assumptions about f, profits $(\theta f(n) - \theta f'(n)n)$ need not be 0. We could give the profits out in an arbitrary way, since agents are risk-neutral and have no disutility of labor.

5 Overlapping Generations Model

Definition 7 Fiat money *is money that is printed by the government that is not backed.* Commodity money *is money printed by the government that is backed by some commodity.*

In a real economy, fiat money will never have any value. Fiat money can take on value if:

- money is included in the utility function,
- consumers have a cash-in-advance constraint, or
- there are market frictions (as in the overlapping generations model).

In the overlapping generations model:

- agents live two periods,
- at date t, there are $N(t)$ young people, so that the total population is $N(t)+N(t-1)$ (we also start out with some initial old people at date 0),
- agents can only trade with living agents,
- it is an endowment economy in which the young get y and the old get ϵ, with $y > \epsilon$, and
- all agents have preferences $\log(c_{young}) + \log(c_{old})$.

In a steady state in the economy, it is not necessary that agents consume the same amount each period. However, all of the agents must consume the same amount in youth and old age. That is, all the consumption paths are the same.

Definition 8 *In this context, an equilibrium is* stationary *if $c_t(t) = c_y$ and $c_t(t+1) = c_o$ for all $t \geq 1$. (There are no restrictions on $c_0(1)$.) Equivalently, an equilibrium is stationary if the return (R_t or $\frac{p(t)}{p(t+1)}$) is constant over time.*

In the competitive equilibrium outcome, there will be *autarky*, where each individual consumes their endowment. Though the old would like to borrow from the young (to smooth consumption), there is no way to pay back the loans because of the finite lifespans. In this equilibrium, the interest rate, $1+r$ is the slope of the indifference curve at the endowment. (Though there is no borrowing, this is the interest rate where you would save the current amount if it were an option.)

The social planner outcome would have each person consume $\frac{y+\epsilon}{2}$ in each period. At this solution, everyone is better off as long as the world never ends (otherwise, the young in the final period T would lose out). Because there cannot be saving, there is *dynamic inefficiency*. To fix this, one could introduce an infinitely-lived government.

One option is pay-as-you-go pensions or Social Security. If the transfer in this case is high enough, then fiat money will no longer be necessary; that is, there will be no equilibrium with fiat currency.

Definition 9 *The* Balasko-Shell condition for optimality *is that* $1+r \geq 1+g$*, where* g *is the growth rate of the economy. If this condition holds, the economy is Pareto optimal.*

In this economy, $r < 0$ so the economy is inefficient. If we had $\epsilon \geq y$, the economy would be efficient. There would still be no trading, but it would be optimal because there is no way to make the people who are old at time 0 better off.

By Samuelson's two proposals, pay-as-you-go Social Security and fiat money would both make the economy optimal, and would be equivalent.

Definition 10 *An* equilibrium with valued fiat money *is a feasible allocation,* $(c_t(t), c_t(t+1))_{t=1}^{T}$*, and a nominal price sequence,* $\{p(t)\}$*, with* $0 < p(t) < \infty$ *for all* t*, such that given the price sequence, the allocation maximizes each households utility subject to the budget constraint* $c_t(t) + c_t(t+1) \leq w_t(t) + \frac{p(t+1)}{p(t)} w_t(t+1)$.

The household's budget constraint can be written in two-period form as:

$$\begin{aligned} c_t(t) + \frac{m(t)}{p(t)} &\leq w_1 \\ c_t(t+1) &\leq w_2 + \frac{m(t)}{p(t+1)} \end{aligned}$$

The initially old have the one-period constraint that $c_0(1) \leq \frac{m(1)}{p(1)} + w_2$.

In the more general case of this model, let

- $w_t^h(s)$ be the endowment of agent h, born at time t during period s,
- $c_t^h(s)$ be the consumption of agent h, born at time t during period s,
- $\tau_t^h(s)$ be the lump sum tax on agent h, born at time t during period s,

- $\tau_t^h(s)$ be the storage of the good by agent h, born at time t during period s (note that this will only happen in period t, so that the agent will be around to take it out of storage),
- $l_t^h(s)$ be the amount of 1-period loans that in period s made by agent h born in time t (this will only be non-zero when the agent is young, so that they can get paid back),
- $G(t)$ be government expenditures at time t,
- $L^g(t)$ be government loans at time t,
- $\rho(t)$ be the net rate of return on storage from period t to $t+1$, and
- $r(t)$ be the net real interest rate between periods t and $t+1$.

Time starts at period 1 and ends at period T (which may be infinite). The initial old have preferences $u_0^h(c_0^h(1))$ while an agent born in period $t \geq 1$ has preferences given by $u_t^h(c_t^h(t), c_t^h(t+1))$. The storage technology takes in $k_t^h(t)$ at time t and returns $k_t^h(t)(1+\rho(t))$ in the next period. The population is $N(t) + N(t-1)$ at each period.

Since the social planner solution is not necessarily implementable, we use compute the competitive equilibrium directly by finding agents' solutions for any prices and then choosing prices by ensuring that markets clear. The agents' problem is to maximize $u_t^h(c_t^h(t), c_t^h(t+1))$ subject to:

$$\begin{aligned} c_t^h(t) + \tau_t^h(t) + l_t^h(t) + k_t^h(t) &\leq w_t^h(t) \\ c_t^h(t+1) + \tau_t^h(t+1) &\leq w_t^h(t+1) + k_t^h(t+1)(1+\rho(t)) + l_t^h(t)(1+r(t)) \\ c_t^h(t), c_t^h(t+1), k_t^h(t) &\geq 0 \end{aligned}$$

This can also be written as a present value budget constraint:

$$c_t^h(t) + \tau_t^h(t) + \frac{c_t^h(t+1)}{1+r(t)} + \frac{\tau_t^h(t+1)}{1+r(t)} \leq w_t(h) + \frac{1_t^h(t+1)}{1+r(t)} + k_t^h(t)(\frac{1+\rho(t)}{1+r(t)} - 1)$$

Note that if $\rho(t) > r(t)$, the agents will set $k_t^h(t)$ to be infinite, which requires that $\rho(t) \leq r(t)$. (The non-negativity constraint on $k_t^h(t)$ allows $\rho(t) < r(t)$ without making the budget constraint unbounded, since the agents cannot "go short" in storage.) If there is no storage, $\rho(t) = -1$. The government budget constraint is:

$$G(t) + L^g(t) = \sum_h \tau_{t+1}^h(t) + \sum_h \tau_t^h(t) + L^g(t+1)(1+r(t-1))$$

(In equilibrium, the government has no reason to store, since they can do at least as well in the capital market.) We assume that $L^g(0) = l_0^h(0) = k_0^h(0) = 0$.

Definition 11 *Given the exogenous components,* $w_0^h(1), N(0), (w_t^h(t), w_t^h(t+1), \rho(t), G(t), N(t))_{t=1}^T$, *a* rational expectations competitive equilibrium *consists of an allocation,* $c_0^h(1), \tau_0^h(1), (c_t^h(t), c_t^h(t+1), k_t^h(t), l_t^h(t), L^g(t), \tau_t^h(t), \tau_t^h(t+1))_{t=1}^T$ *and a price system,* $r(t)_{t=1}^T$*, that satisfy:*

- *the government budget constraint,*
- *given* $(r(t), \rho(t), \tau_t^h(t), \tau_t^h(t+1))$*, agents are maximizing utility, and*
- *market-clearing conditions:*

$$\begin{aligned} \sum_h c_{t-1}^h(t) + \sum_h c_t^h(t) + G(t) + \sum_h k_t^h(t) &= \sum_h w_{t-1}^h(t) + \sum_h w_t^h(t) + \sum_h k_{t-1}^h(t-1)(1+\rho(t-1)) \\ L^g(t) + \sum_h l_t^h(t) &= 0 \end{aligned}$$

A sequence of $L^g(t)$ is equivalent to having fiat money. Suppose that the government prints bills and gives them to the old. It is possible that people will assign no value to the fiat money, which reduces to the case where $L^g(t) = 0$. However, the fiat money can take on a range of values, which means that the price level is not uniquely determined; any of the possibilities in the range above leads to a rational expectations equilibrium. Now, we see unbacked government debt as fiat money, with the real money supply defined by $-L^g(t) = \frac{m(t)}{p(t)}$ and the (inverse of the) inflation rate by $1+r(t) = \frac{1/p(t+1)}{1/p(t)} = \frac{p(t)}{p(t+1)}$. Substituting these into the government budget constraint leads to:

$$G(t) = \sum_h \tau_{t-1}^h(t) + \sum_h \tau_t^h(t) + \frac{1}{p(t)}(m(t) - m(t-1))$$

The equilibrium conditions now include both debt and money as assets, and they must have the same rate of return. In the case with no taxes after the initial tax and no government spending, the expression above implies that $m(t+1) = m(t)$, so that the nominal money supply is constant but the real money supply changes. In all cases, $L^g(1) = \sum_h \tau_0^h(1) = \frac{-m(1)}{p(1)}$. In this monetary equilibrium, the government controls $m(1)$ but not $p(1)$, because that depends on the agents' perceptions.

In general, to solve a fiat money problem:

- Find the saving function (which is equivalent to the demand for money) by maximizing $u(w_1 - s, w_2 + Rs)$ over s.
- In equilibrium, $s(R_t) = \frac{m}{p(t)}$ and $R_t = \frac{p(t)}{p(t+1)}$. Given $p(1)$ or R_0, these equations implicitly define the path of $p(t)$.

For a stationary equilibrium, we must have $p(t) = p(t+1)$. Otherwise, for equilibrium to occur, $p(t) < p(t+1)$ and R_t decreases to R^* such that $S(R^*) = 0$; this is the interest rate at the equilibrium without valued fiat currency. Any equilibrium must have $R_t \in [R^*, 1]$.

If money is valued, the stationary equilibrium is not stable.

5.1 Example: Borrowers and Lenders

Suppose everyone has utility function $u_t^h(c_t^h(t), c_t^h(t+1)) = \log c_t^h(t) + \log c_t^h(t+1)$. Suppose that in each generation, there are N_1 people with endowment $(\alpha, 0)$ and N_2 people with endowment $(0, \beta)$. We assume that $\rho(t) = -1, T = \infty, G(t) = 0, \tau_t^h(t) = \tau_t^h(t+1) = 0$. We allow $\tau_t^h(0)$ and $L^g(t)$ to take on any values.

Solving the agents' problem shows that $l_t^1(t) = \frac{\alpha}{2}$ and $l_t^2(t) = -\frac{\beta}{2(1+r(t))}$. Using the market clearing condition, we find that $1 + r(t) = \frac{\beta N_2}{\alpha N_1 + 2L^g(t)}$. (This is also equivalent to the feasibility condition.) The government budget constraint implies that:

$$\begin{aligned} L^g(t+1) &= (1 + r(t))L^g(t) \\ L^g(1) &= \sum_h \tau_0^h(1) \end{aligned}$$

Clearly, one solution sets $\tau_0^h = 0$. Then, $L^g(t) = 0$ for all time. In this equilibrium, $1 + r(t) = \frac{N_2\beta}{N_1\alpha}$.

Suppose $\frac{N_2\beta}{N_1\alpha} \geq 1$. Then, there is a continuum of equilibria which can be ranked according to Pareto optimality.

Suppose $1+r(t) = 1$ and therefore $L^g(t) = L^g(1)$ for all t. Then, $L^g(t) = \frac{1}{2}(N_2\beta - N_1\alpha) < 0$. This equilibrium transfers from the young to the old. In general, any $L^g(1) \in [\frac{1}{2}(N_2\beta - N_1\alpha), 0]$ leads to an equilibrium, in which the government borrows and transfers to the old. For the interior cases, $\frac{N_2\beta}{N_1\alpha} < 1 + r(t) < 1$. Then, $0 > L^g(t+1) > L^g(t)$, which means that $1 + r(t+1) < 1 + r(t)$. In the limit, $(L^g(t), 1 + r(t)) \to (0, \frac{N_2\beta}{N_1\alpha})$.

If $\frac{N_2\beta}{N_1\alpha} \geq 1$, then the economy is dynamically efficient (by Balasko-Shell). Thus, the equilibrium above is the only equilibrium. Using the results above, we would see that any debt would grow unboundedly, with $N_1\frac{\alpha}{2} - N_2\frac{\beta}{2(1+r(t))} = -L^g(t)$. Since the left-hand side is bounded by $N_1\frac{\alpha}{2}$ and the right hand side is unbounded, there can be no equilibrium.

Given $p(1)$, the credit market pins down all future $p(t)$ through the difference equation:

$$\begin{aligned} N_1\frac{\alpha}{2} - N_1\frac{\beta}{2}\frac{p(t+1)}{p(t)} &= \frac{m(1)}{p(t)} \\ N_1\alpha p(t) - N_2\beta p(t+1) &= 2m(1) \\ (1 - \frac{N_2\beta}{N_1\alpha}L^{-1})p(t) &= \frac{2m(1)}{N_1\alpha} \end{aligned}$$

Since $\frac{N_2\beta}{N_1\alpha} < 1$, the general solution is

$$p(t) = \frac{2m(1)}{n_1\alpha}\frac{1}{1 - N_2\beta/N_1\alpha} + C(\frac{N_1\alpha}{N_2\beta})^t$$

we require that $C \geq 0$, so that the price level is always positive. Any positive C is possible. $C = \infty$ is equivalent to $p(1) = \infty$ (AND $C = 0$?). This shows that $(\frac{1}{2}(N_2\beta - N_1\alpha), 0)$ maps to $(0, \infty)$.

Contrast this with the case of infinitely lived agents. Then, the government could not have $\tau_o^h < 0$ since $L^g(t) < 0$ for all time, and no one would buy the bonds. Furthermore, $\tau_0^h > 0$ implies that the government would accumulate infinite assets. Infinitely lived agents would have no use for fiat money either. In an Arrow-Debreu equibrium, $p(1) = \infty$; this is the price of money, which is a good that gives no one utility.