Macroeconomics Summary

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To describe an economic model, we require:

- The number of agents
- The number of goods (and how they are indexed)
- Preferences for the agents
- Information about the endowments of the agents
- Information about the technology
- The trading mechanism

A competitive equilibrium specifies an allocation and prices, such that the allocation solves each household's problem given the prices and the production solves the firm's problem under those prices.

For more general optimization problems, the general set-up is:

- Choose ...
- to maximize ...
- such that ... or given ...

1 Linear Stochastic Difference Equations

Definition A *stochastic linear difference equation* is given by the model:

$$\begin{aligned} x_0 &\sim Normal(\hat{x}_0, \Sigma_0) \\ x_{t+1} &= Ax_t + Cw_{t+1} \\ y_t &= Gx_t \\ w_t &\sim_{iid} Normal(0, I) \end{aligned}$$

for all $t \geq 0$. In this model, we call x_t the *state* (and assume that it is not observed), w_t the *shocks*, and assume that y_t is observed.

Note that x_t is a Markov process, with $x_{t+1}|x_t \sim Normal(Ax_t, CC^T)$.

To compute moments of x_t, we note that

$$
\begin{aligned}
x_t &= A^t x_0 + A^{t-1}Cw_1 + A^{t-2}Cw_2 + ... + ACw_{t-1} + Cw_t \\
\mu_{x_t} &= E(X_t) = A^t \hat{x}_0 \\
\Sigma_{x,t} &= A\Sigma_{x,t-1}A^T + CC^T
\end{aligned}
$$

Note that $\mu_{x_{t+1}} = A\mu_{x_t}$, which is a non-stochastic linear difference equation for the mean.

To be able to use these equations to estimate moments from data, we must impose covariance stationarity, so that μ_{y_t} and therefore μ_{x_t} are constant. Suppose we may write:

$$
\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} + \begin{pmatrix} 0 \\ C_2 \end{pmatrix} w_{t+1}
$$

(Note that the first term corresponds to a non-zero mean.) Suppose that all of the eigenvalues of A_{22} have modulus strictly less than 1. Then, we choose the mean, μ_x, to be the eigenvector of A associated with the unit eigenvalue. If, instead, A has all of its eigenvalues strictly less than 1 (in modulus), we choose $\mu_x = 0$. For covariance stationarity, we must choose $\hat{x}_0 = \mu_x$. In addition, for covariance stationarity, we must have $\Sigma_{x,t}$ constant. This yields the *discrete Lyapunov equation*, $\Sigma_x = A\Sigma_x A^T + CC^T$ (which can be solved by guessing any $\Sigma^{(0)}$ and iterating on $\Sigma_x^{(n)} = A\Sigma_x^{(n-1)}A^T + CC^T$ until it converges).

We may also compute auto-covariances of x and y (assuming covariance stationarity, for simplicity):

$$
\begin{aligned}
C_x(j) &= E((x_{t+j} - \mu_x)(x_t - \mu_x)^T) \\
&= A^j \Sigma_x \\
C_y(j) &= GC_x(j)G^T \\
&= GA^j \Sigma_x G^T
\end{aligned}
$$

This yields a non-stochastic difference equation for the covariances as well.

We may also compute conditional expectations:

$$
\begin{aligned}
E(x_{t+j}|x_t) &= A^j x_t \\
E(y_{t+j}|x_t) &= GA^j x_t
\end{aligned}
$$

We may transform an $ARMA(p,q)$ model for y_t into this form by setting $x_t = (y_t, y_{t-1}, ..., y_{t-p}, w_{t-1}, ..., w_{t-q})^T$, setting the first row of A equal to the coefficients (and the other rows equal to 0's and 1's so that the lags are moved back one period), C to a vector of 0's and 1's that adds w_t to x_t and possibly keeps in the information set, and $G = (1, 0, ..., 0)^T$.

If this model comes from a linear approximation to some other model, A, C, and G will be functions of some *deep parameters*, θ. Then, we will be able to compute moments based on the value of θ (or estimate θ from the moments based on data).

1.1 Vector Autoregression

Definition Assume that $E(y_t) = 0$ and that y_t is covariance stationary. A *vector autoregression (VAR)* is an equation of the form:

$$y_t = \sum_{j=1}^{\infty} A_j y_{t-j} + a_t$$

where $E(a_t y_{t-j}^T) = 0$ for all $j \geq 1$. An *vector autoregression of order N (VAR(N))* is given by:

$$y_t = \sum_{j=1}^{N} A_j^{(N)} y_{t-j} + a_t^{(N)}$$

and imposes only $E(a_t^{(N)} y_{t-j}^T) = 0$ for $j = 1, ..., n$. In these equations, a_t is the *linear least squares forecast error.*

Note that this is a population equation (these are not estimates).

To calculate A_j, we minimize $trace(E((y_t - \sum_{j=1}^N A_j^{(N)} y_{t-j})(y_t - \sum_{j=1}^N A_j^{(N)} y_{t-j})^T))$. This yields the first order conditions (also called the normal equations):

$$\begin{aligned} 0 &= E\left(\left(y_t - \sum_{j=1}^{N} A_j^{(N)} y_{t-j}\right) y_{t-k}^T\right) \\ &= C_y(j) - \sum_{j=1}^{N} A_j^{(N)} C_y(k-j) \end{aligned}$$

This is a population version of least squares.

Calculation gets cumbersome as $N \to \infty$.

1.2 State Space Models and the Kalman Filter

Suppose we have a stochastic linear difference equation with measurement error in y_t:

$$\begin{aligned} x_0 &\sim Normal(\hat{x}_0, \Sigma_0) \\ x_{t+1} &= Ax_t + Cw_{t+1} \\ y_t &= Gx_t + v_t \\ w_t &\sim_{iid} Normal(0, I) \\ v_t &\sim_{iid} Normal(0, R) \end{aligned}$$

where v_s and w_t are independent for all s, t. Note that this model has $m + p = dim(w) + dim(v)$ shocks in each period, but we observe only p variables in each period. In a VAR for y_t, the p shocks in $a_t^{(\infty)}$ depend on both v_t and w_t.

If we could observe x_t, then we would have the likelihood function:

$$\begin{aligned} L(\{x_t\}_{t=0}^T) &= f(x_T|x_{T-1})...f(x_1|x_0)f(x_0) \\ l(\{x_t\}_{t=0}^T) &= \log f(x_0) + \sum_{t=1}^T \left(-\frac{1}{2}(x_t - Ax_{t-1})(CC')^{-1}(x_t - Ax_{t-1})' - \frac{1}{2}\log det(CC') \right) \end{aligned}$$

(where $f(x_0)$ is the density for a normal distribution with mean $\hat{x}_0$ and variance Σ_0).

However, y_t is not Markov. Because we cannot condition on x_t, we use our observations of y_t (and our knowledge of $\hat{x}_0$) to estimate x_t by $\hat{x}_t$. That is, $\hat{x}_t = E(x_t|y_{t-1}, ..., y_0, \hat{x}_0)$. Then, we have the system:

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + K_t a_t \\ y_t &= G\hat{x}_t + a_t \end{aligned}$$

where $a_t = y_t - E(y_t|y_0, ..., y_{t-1}, \hat{x}_0)$ (the innovation in y_t) and $K_t = A\Sigma_t G'(G\Sigma_t G' + R)^{-1}$.

Note that

$$\Sigma_{t+1} = A\Sigma_t A' + CC' - A\Sigma_t G'(G\Sigma_t G' + R)^{-1} G\Sigma_t A'$$

where Σ_0 comes from the initial density. This is a *Ricatti difference equation*.

For a properly chosen Σ_0, Σ_t and therefore K_t will be constant; this is equivalent to seeing the infinite past of y. Then, $a_t = y_t - E(y_t|y_{t-1}, ..., y_{-\infty})$ and the a_t will equal the VAR errors. That is, we have a recursive representation of the VAR, also called the *innovations representation of* y_t:

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + K a_t \\ y_t &= G\hat{x}_t + a_t \end{aligned}$$

Also, note that K_t can be interpreted as the coefficient from the regression of x_{t+1} on a_t. Since the a_t are orthogonal to each other by assumption, the coefficients of the regression of x_t on $a_{t-1}, ..., a_0, \hat{x}_0$ would be unchanged by including a_t. Thus, the updating method for $\hat{x}_t$ is like adding another regressor. The algorithm for computing $\hat{x}_t$ is called the *Kalman filter*.

This is equivalent to minimizing $\sum_{t=1}^n (Y_t - \hat{Y}_t)^2$ subject to the constraint $Y = GX + \epsilon\Sigma, X = HX_{-1} + V\Omega$ by choosing G, Σ, Ω, H.

Then, we may write the likelihood recursively as:

$$f(y_T|\hat{x}_T)f(y_{T-1}|\hat{x}_{T-1})...f(y_1|\hat{x}_1)f(y_0)$$

where the last term will depend on the stationary distribution, and $y_t|\hat{x}_t \sim Normal(G\hat{x}_t, G\Sigma_t G' + R)$. (This is used in dynamic stochastic general equilibrium models.)

More observations will improve the estimates of X, since one can improve the fit each period.

1.3 Linear-Quadratic Programming with No Discounting

Suppose we want to maximize $-\sum_{t=0}^{\infty}(x_t'Rx_t + u_t'Qu_t)$ subject to $x_{t+1} = Ax_t + Bu_t$ (that is, x_t is the state and u_t is the control). This is a *linear-quadratic model with no discounting.* Let $v(x_0) = -x_0'Px_0$ be the optimal value function. Then,

$$-x'Px = \max_u(-(x'Rx + u'Qu) - (x^*)'Px^*)$$

where $x^* = Ax + Bu$. Solving this, we find that:

$$\begin{aligned} u &= -Fx \\ F &= (Q + B'PB)^{-1}B'PA \\ P &= R + A'PA - A'PB(Q + B'PB)^{-1}B'PA \end{aligned}$$

(If we did not impose that $P_t = P_{t+1}$, then we would have $P_t = R + A'P_{t+1}A - A'P_{t+1}B(Q + B'P_{t+1}B)^{-1}B'PA$.)

This can be mapped to a Kalman filter, but time runs in the opposite direction. In fact, the Kalman filter problem is the dual of the dynamic programming problem, so every linear-quadratic Bellman equation maps to a Kalman filter. (There is a more general duality as well, but it does not extend to all problems.)

A linear-quadratic model with no discounting can also be solved with a Lagrangian:

$$L = -\sum_{t=0}^{\infty}(x_t'Rx_t + u_t'Qu_t + 2\mu_{t+1}'(Ax_t + Bu_t - x_{t+1}))$$

Note that there is one constraint for each period. This leads to the first order conditions (with respect to u_t, x_t, and μ_t) for each $t \geq 0$:

$$\begin{aligned} 2Qu_t + 2B'\mu_{t+1} &= 0 \\ u_t &= Rx_t + A'\mu_{t+1} \\ x_{t+1} &= Ax_t + Bu_t \end{aligned}$$

We can then solve for one period in terms of the previous period:

$$L\begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = N\begin{pmatrix} x_t \\ \mu_t \end{pmatrix}$$

We call and then apply the Schur Decomposition.

This is another way to solve the Bellman equation, with the help of the correct shadow price.

2 Analyzing Dynamic Systems

Suppose one can write $Cx_{t+1} = Bx_t$, or, equivalently, $x_{t+1} = Ax_t$. Then, $x_t = A^t x_0 = V\Sigma^t V^T x_0$, where $A = V\Sigma V^T$ is the eigenvalue decomposition.

The elements of x_0 must be chosen to match the initial conditions and the terminal conditions (stability in the limit, positivity for all time, or something else).

To study a dynamic system:

1. Find the steady state.
2. Study the eigenvalues and eigenvectors.

To deal with terminal conditions, one must compute with possible values of x_0 until one works.

Shooting Algorithm:

- Choose c_0.
- Use c_0 and k_0 to compute the entire path of (c_t, k_t).
- If the path does not converge, adjust c_0.

2.1 Schur Decomposition

Suppose we have:

$$L \begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = N \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}$$

We call

x_t the *state variables* and μ_t the *co-state variables.* If L is invertible (though this would all work out if it weren't), we have $\begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = M \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}$, where $M = L^{-1}N$. This leads to the difference equation system $\begin{pmatrix} x_{t+1} \\ \mu_{t+1} \end{pmatrix} = M^{t+1} \begin{pmatrix} x_0 \\ \mu_0 \end{pmatrix}$, where x_0 is a given initial condition. Note that if $x_t \to \pm\infty$ then $x_t R x_t \to \infty$ and the value function is unboundedly negative. Thus, we must choose μ_0 to ensure that x_t does not diverge. (These are called *transversality conditions.*)

Proposition 2.1 *Since M came from the first order conditions of an undiscounted infinite horizon optimization problem, if λ is an eigenvalue of M, then so is $\frac{1}{\lambda}$.*

Proof (Sketch.) M is a symplectic matrix, so $MJM' = J$ when $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. Then, $M' = J^{-1}M^{-1}J$, and M' has the same eigenvalues as M^{-1}. Since M and M' also have the same eigenvalues, the eigenvalues must come in reciprocal pairs. ∎

We may modify R, Q to ensure that no eigenvalues are of modulus exactly 1, so half of the eigenvalues have modulus less than 1 and the other half have modulus greater than 1.

Let $y_t = \begin{pmatrix} x_t \\ \mu_t \end{pmatrix}$, so that we are trying to solve $y_t = M^t y_0$. Applying the *Schur decomposition*, there is some V such that $V^{-1}MV = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix} = W$, with all the eigenvalue of W_{11} less than 1 in modulus and all the eigenvalues of W_{22} greater than 1 in modulus. Then, $y_{t+1} = VWV^{-1}y_t$, or $y^*_{t+1} = V^{-1}y_{t+1} = WV^{-1}y_t = Wy^*_t$. Then, $y^*_t = W^t y_0 = \begin{bmatrix} W_{11}^{t+1} & W_{12,t+1} \\ 0 & W_{22}^{t+1} \end{bmatrix} y_0$, where $W_{12,t} = W_{11}^{t-1}W_{12} + W_{12}W_{22}^{t-1}$. Note that $W_{11}^t \to 0$ and W_{22}^t diverges because of their eigenvalues. Thus, we require that $y^*_{2,0} = 0$ to prevent divergence. This is equivalent to requiring that $V_{21}x_0 + V_{22}\mu_0 = 0$. Let $V^{-1} = \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}$. Then, $\mu_0 = -(V^{22})^{-1}V^{21}x_0$. Using a partitioned inverse formula, we find that $\mu_0 = V_{21}V_{11}^{-1}x_0$. This yields an initial condition for μ_0 that ensures the stability of the system and will lead to the optimal solution. Note that for all $t \geq 0$, $\mu_t = V_{21}V_{11}^{-1}x_t$. Furthermore, $-2\mu_t$ is the shadow price of x_t, which is equal to $\frac{\partial V}{\partial x_0}(x_0) = -2Px_0$. Thus, $Px_0 = V_{21}V_{11}^{-1}x_0$.

2.2 Linear Approximation

Suppose we have a model with forcing variables z_t and endogenous variables, k_t, related by an equation, $H(k_t, k_{t+1}, z_t, z_{t+1}, z_{t+2}) = 0$. Assume that $z_t = \overline{z}$ for all $t > T$ for some T.

First, solve for the steady state, using $H(\bar{k}, \bar{k}, \bar{z}, \bar{z}, \bar{z}) = 0$.

Second, apply a Taylor series approximation about the steady state :

$$H_{k_t}(k_t - \bar{k}) + H_{k_{t+1}}(k_{t+1} - \bar{k}) + H_{z_t}(z_t - \bar{z}) + H_{z_{t+1}}(z_{t+1} - \bar{z}) + H_{z_{t+2}}(z_{t+2} - \bar{z}) = 0$$

(where subscripts indicate partial derivatives with respect to that variable evaluated at the steady state). This is a second order linear difference equation and can be written as:

$$\begin{aligned} \phi_0 k_{t+2} + \phi_1 k_{t+1} + \phi_2 k_t &= A_0 + A_1 z_t + A_2 z_{t+1} \\ \phi(L) k_{t+2} &= A_0 + A_1 z_t + A_2 z_{t+1} \end{aligned}$$

We factor $\phi(L) = \phi_0(1 - \lambda_1 L)(1 - \lambda_2 L)$. For "most" problems, $|\lambda_2| < \frac{1}{\sqrt{\beta}}$ and $|\lambda_1| > \frac{1}{\sqrt{\beta}}$; for now, we assume that $|\lambda_2| < 1$ and $|\lambda_1| > 1$. Then, we may write:

$$\begin{aligned} (1 - \lambda_2 L)^{-1} &= \sum_{j=0}^{\infty} \lambda_2^j L^j \\ (1 - \lambda_1 L)^{-1} &= \left(-\lambda_1 L (1 - \frac{1}{\lambda_1 L}) \right)^{-1} \\ &= -\frac{1}{\lambda_1} L^{-1} \sum_{j=0}^{\infty} \frac{1}{\lambda_1^j} L^{-j} \end{aligned}$$

Thus, one root has a stable forward inverse and the other has a stable backward inverse. We may rewrite the difference equation as:

$$\begin{aligned}
-\lambda_1^{-1}\phi_2(1-\lambda_1^{-1}L^{-1})(1-\lambda_2 L)Lk_{t+2} &= A_0 + A_1 z_t + A_2 z_{t+1} \\
(1-\lambda_2 L)k_{t+1} &= \frac{-\lambda_1}{\phi_2(1-\lambda_1^{-1}L^{-1})}(A_0 + A_1 z_t + A_2 z_{t+1}) \\
k_{t+1} - \lambda_2 k_t &= -\frac{\lambda_1}{\phi_2}\left(\frac{A_0}{1-\lambda_1^{-1}} + A_1\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t+j} + A_2\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t+j+1}\right) \\
k_{t+1} &= \lambda_1 k_t - \frac{\lambda_2}{\phi_2}\left(\frac{A_0}{1-\lambda_1^{-1}} + A_1\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t+j} + A_2\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t+j+1}\right)
\end{aligned}$$

This will be the solution that satisfies the boundary conditions.

If there is a one-time change at time T, then the foresight of the change corresponds to the terms in $-\frac{\lambda_1}{\phi_2}(\frac{A_0}{1-\lambda_1^{-1}} + A_1\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t+j} + A_2\sum_{j=0}^{\infty}\lambda_1^{-j}z_{t+j+1})$ and the returns to the steady state corresponds to movement with $k_{t+1} - \lambda_2 k_t = 0$. This occurs because the sums change as the upcoming shift(s) moves closer and then are constant after the shift(s).

If the steady state does not change and if there is no anticipation of a policy change, then there will be no transition at all.

Most of the computational methods use approximations. If there is a very large change, then the approximation might fail. Also, in simulation, one might get close to the steady state and then veer off.

2.3 Dynamics with eigenvectors (and pictures)

For any model, we may compute $c(k)$ such that $k_t = k_{t+1}$ when consumption is set at $c(k)$. At any point above the curve, capital will decrease in the next period, since there is too much consumption to keep capital constant. (This is a tendency to move left.) At any point below the curve, capital will increase (the point moves right).

The steady state capital leads to a vertical line of where capital will eventually be. If capital lies above the steady state, then consumption must decrease in the future (the point moves down). If current capital lies below the steady state, consumption increases in the future (the point moves up).

The intersection of the two curves determines the steady state, while the location relative to the curves determines the dynamics of how to get there. Many paths diverge; usually only one leads back to the steady state. With an anticipated change, the transition is always onto the path to a steady state. (First, one jumps from the old steady state to the new trajectory. Then, one moves also the new trajectory to the new steady state.)

With multiple changes, steady state capital may shift multiple times (there might not be time to get to the steady state, but we can imagine the steady state capital moving anyway). This will affect consumption and capital choices in the next period. After all the changes, consumption and capital will move monotonically back to the steady state.

The direction of motion depends on the eigenvectors. Movement along the eigenvector is toward the steady state if the corresponding eigenvalue is less than 1 in modulus and away from the steady state if the corresponding eigenvalue is greater than 1 in modulus. For stability, all non-zero elements in the first period are associated with eigenvalues less than 1 in modulus. This is not always possible.

If all eigenvalues are less than 1, then there may be multiple solutions (indeterminacy) and any point will lead to some steady state. If all eigenvalues are greater than 1 in modulus, then the equilibrium is unstable, and any movement away from it will lead to divergence. The eigenvalue with the largest modulus will dominate the long run behavior, with $x_t \approx \lambda_K^t v_K c_K$, where λ_K is the largest eigenvalue, v_K is the associated eigenvalue, and c_K is the initial value. If $c_K = 0$, a smaller eigenvalue will dominate, but that is unstable.

3 The Competitive Equilibrium Model

Definition In this model, we define the *state* as $s_t \in S$, where S does not depend on time. The *history* of an economy is $s_0^t = s^t = \{s_0, s_1, ..., s_t\}$. Note that s_t is a random variable and s^t is a stochastic process. We define $\pi_t(s^t)$ to be the probability density of s^t.

3.1 Arrow-Debreu

We first consider a pure exchange economy with agents $i = 1, ..., I$.

- The *endowment* of an individual at a certain time after a certain history is given by $y_t^i(s^t)$. We assume that the endowment is exogenous.
- The technology is *pure endowment*, which means that the good is non-storable and that the aggregate endowment at time t is $\sum_{i=1}^I y_t^i(s^t)$.
- Let $c^i = \{c_t^i(s^t)\}_{t=0}^\infty$ be the stochastic process of consumption. We define *preferences* on consumption by $U(c^i) = \sum_{t=0}^\infty \sum_{s^t \in S^t} \beta^t u^i(c_t^i(s^t))\pi_t^i(s^t)$. $u^i(c)$ is the one period utility function, assumed to be increasing, concave, twice continuously differentiable, and with $u'(0) = \infty$ (we often assume that $u^i = u$ is the same for all individuals). $\pi_t^i(s^t)$ reflects the subjective beliefs of the individual about the stochastic process governing the economy (we usually assume that $\pi_t(s^t)$ is constant across all individuals and is the true process in the economy).
- We have *Arrow-Debreu trading*, where there are complete markets for all state-contingent goods at time 0. In these markets, $q_t^0(s^t)$ is the *history-date price* (also called the *time zero price*) of one unit of consumption at date t, if history s^t occurs. (This leads to $|S|^t$ prices at time t, one for each history.)

Preferences depend on expected utility. Von Neumann assumed that the true π_t was known to everyone. Savage allowed π_t^i to vary, which gives Bayesian or subjective expected utility. Muth simply equated π_t across all individuals, the true process in nature, and the econometrician. This is then exploited to do rational expectations econometrics (and reduces the number of parameters in the models). This might be plausible if agents have all been observing the same economy over a long time, and therefore are all approximately correct because of the law of large numbers.

Definition Two probability distributions, $\pi_t^i(s^t)$ and $\pi_t^j(s^t)$ are *mutually absolutely continuous* if $\pi_t^i(s^t) = 0$ if and only if $\pi_t^j(s^t) = 0$. (That is, they consider the same events possible, though the probabilities might differ across possible events.)

If the subjective probability distributions are not mutually absolutely continuous, then there is some $\tilde{s}^t \in S^t$, and some individuals i, j such that $\pi_t^i(\tilde{s}^t) = 0$ and $\pi_t^j(\tilde{s}^t) > 0$. Then, the price for the goods in that state is $q_t^0(\tilde{s}^t) > 0$, and i will sell not only his endowment $y_t^i(\tilde{s}^t)$ but would like to go short and sell more than his endowment; this means that no equilibrium exists (and we might need to impose a restriction like $c_t^i \geq 0$ to ensure an equilibrium). This would not be possible if the probability were positive, because $u'(0) = -\infty$ would ensure that 0 or negative consumption is never chosen for any event with positive probability. Mutually absolute continuity imposes the constraint that beliefs about tail events (which can only be 0 or 1) must be identical.

Definition In this economy, the *present value of an agent's endowment* if given by $\sum_{t=0}^{\infty} \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$. (Note that because the prices are set and known at time 0, there is no need for discounting or including probabilities.) The *present value of an agent's consumption* is $\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t)$. Prices are determined endogenously.

Definition *Financial wealth* is the present value of future claims, $\sum_{t=0}^{\infty} \sum s^t q_0^t(s^t)(c_t^i(s^t) - y_t^i(s^t))$. *Human wealth* is the present value of future (labor) income.

The *agent's problem* in this economy is to maximize $U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u^i(c_t^i(s^t)) \pi_t^i(s^t)$ subject to the constraint $\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)(c_t^i(s^t) - y_t^i(s^t)) \leq 0$, given $\{q_t^o(s^t)\}$. That is, they sell off their endowment and use the profits to choose consumption across states and times. There are no expectations in the budget constraint, only in the utility maximization.

Definition A *feasible allocation* is a vector of $c_t^i(s^t) \geq 0$ that satisfies $\sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y_i^t(s^t)$ for all t, s^t.

Definition A *price system* is a stochastic process, $q_t^0(s^t)$, which is measurable with respect to s^t (that is, the price at time t is known if the history, s^t, is known).

Definition As *Arrow-Debreu competitive equilibrium* is a price system and a feasible allocation such that, given the price system, the allocation is the solution to the household problem for each $i = 1, ..., I$.

That is, the households solve their problem at time 0, knowing prices, future endowments, probabilities, and their own preferences. They trade in every market (since $u'(0) = \infty$). We assume there is an "auctioneer" outside the model that sets prices and ensures that each household satisfies the budget constraint. After time 0, no more trades are made; only deliveries are made.

To compute an equilibrium, we first consider the social planner's problem in a command economy. Suppose each individual has a *Pareto weight*, $\lambda_i > 0$. The social planner has a welfare function, $W = \sum_{i=1}^{I} \lambda_i U(c^i)$. The Pareto problem is to maximize W subject to $\sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y_i^t(s^t)$ for all t, s^t. (This has no enforcement or information problems.) We use Lagrange multipliers:

$$\begin{aligned} L &= \sum_{i=1}^{I} \lambda_i U(c^i) + \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \theta_t(s^t) \left(\sum_{i=1}^{I} y_t^i(s^t) - \sum_{i=1}^{I} c_i^t(s^t) \right) \\ &= \sum_{i=1}^{I} \lambda_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) + \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \theta_t(s^t) \left(\sum_{i=1}^{I} y_t^i(s^t) - \sum_{i=1}^{I} c_i^t(s^t) \right) \end{aligned}$$

Applying the Kuhn-Tucker Theorem, we maximize over the c^i and minimize over the $\theta_t(s^t)$. This gives the first order necessary conditions for $c_t^i(s^t)$ for all i, t, s^t:

$$\lambda_i \beta^t u'(c_t^i(s^t)) \pi_t - \theta_t(s^t) = 0$$

The complementary slackness conditions are:

$$\theta_t(s^t) \left(\sum_{i=1}^{I} (y_t^i(s^t) - c_t^i(s^t)) \right) = 0$$

Using the first order conditions, and noting that $\theta_t(s^t)$, β^t, and $\pi_t(s^t)$ do not depend on i, we find that

$$\frac{u'(c_t^1(s^t))}{u'(c_t^i(s^t))} = \frac{\lambda_i}{\lambda_1}$$

for any $i = 2, ..., I$; the equilibrium ratio of marginal utilities depends only on the Pareto weights, which are time and history invariant. Thus, as λ_i increases, i will consume more uniformly across all times and states. With strict concavity, u' is invertible, and we may solve:

$$c_t^i(s^t) = (u')^{-1}\left(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t))\right)$$

If the allocation uses all the resources, we may substitute this to find:

$$\sum_{i=1}^{I} y_t^i(s^t) = \sum_{i=1}^{I} (u')^{-1} \left(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t)) \right)$$

Since $y_t^i(s^t)$ is known, this is one equation in one variable $(c_t^1(s^t))$ for each t, s^t. Thus, each individual's consumption depends only on the aggregate endowment and the Pareto weights. If there are two histories, $\tilde{s}^t$ and $\overline{s}^t$ such that $\sum_{i=1}^I y_t^i(\tilde{s}^t) = \sum_{i=1}^I y_t^i(\overline{s}^t)$, then consumption is the same in the two histories. This implies that there is no history dependence in consumption.

We can use these results to compute the multipliers, $\theta_t(s^t)$ (the result will be the same for any choice of i). In general, the multipliers will be affected by the choice of Pareto weights.

In a competitive equilibrium, the *household problem* leads to the Lagrangian:

$$L = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u(c_t^i(s^t))\pi_t(s^t) + \mu_i \left(\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)(y_t^i(s^t) - c_t^i(s^t)) \right)$$

Again, we maximize over the $c_t^i(s^t)$ and minimize over the μ_i. This gives the first order necessary condition for $c_t^i(s_t)$ for each t, i, s^t:

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) - \mu_i q_t(s^t) = 0$$

We may match up the first order necessary conditions for the social planner's problem and the household's problem:

$$\begin{aligned} \frac{1}{\mu_i}\beta^t u'(c_t^i(s^t))\pi_t(s^t) &= q_t(s^t) \\ \lambda_i \beta^t u'(c_t^i(s^t))\pi_t(s^t) &= \theta_t(s^t) \end{aligned}$$

In contrast to the planner's problem, the household's endowment implicitly determines $\frac{1}{\mu_i}$, which is their Pareto weight, but depends on unknown prices. This gives a system of simultaneous equations:

$$\begin{aligned} \frac{1}{\mu_i}\beta^t u'(c_t^i(s^t))\pi_t(s^t) &= q_t(s^t) \\ \sum_t \sum_{s^t} q_t^0(s^t)(y_t^i(s^t) - c_t^i(s^t)) &= 0 \end{aligned}$$

which must be solved for $q_t^0(s^t), c_t^i(s^t), \mu_i$ for all i, t, s^t.

Negishi Algorithm

- Choose arbitrary positive Pareto weights. (Without loss of generality, we may assume that $\sum_{i=1}^I \lambda_i = 1$.)
- Compute the allocation using these weights.
- Solve for $\theta_t(s^t)$ and set $q_t^0(s^t) = \theta_t(s^t)$.
- Check the budget constraint, $\sum_t \sum_{s^t} q_t^0(s^t)(y_t^i(s^t) - c_t^i(s^t)) = 0$ for each household. If household i is spending too much, then decrease λ_i. If household i is spending too little, increase λ_i.

- Repeat with the new weights until all the budget constraints are exactly satisfied.

(This becomes computationally intensive for large I, but works for any preferences.)

Definition Under *constant relative risk aversion*, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ (and $u(c) = \ln(c)$ for $\gamma = 1$). Then, $u'(c) = c^{-\gamma}$.

Note that these preferences are *homothetic* (along any ray from the origin, the slope of the indifference curves the ray intersects is constant; this also implies Gorman aggregation and linear Engle curves through the origin). In this case, the marginal rate of substitution (across time and history) depends only on ratios. Also, $\theta_t(s^t)$ does not depend on λ_i. Thus, we may compute the prices based on any λ_i and compute allocations given those prices.

Definition The *state price deflator* is given by $p_0^t(s^t) = q_t^0(s^t)/\pi_t(s^t)$. (This is more common than time zero prices in finance.)

Also in finance, sometimes *scaled Arrow-Debreu prices* are used; these are $p_t^0(s^t) = \frac{q_t^0(s^t)}{\beta^t \pi_t(s^t)}$. Then, we have:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) y_t^i(s^t) = \sum_{t=0}^{\infty} \beta^t E(p_t^0(s^t) y_t^i(s^t))$$

Definition Suppose an asset, z, pays dividends (in terms of the consumption good) $d_t(s^t)$ in time t. Suppose that $q_t^0(s_t)$ is the price of 1 unit of consumption paid at time t and state s^t. Then the *asset pricing formula* for this asset at time 0 is:

$$p_0(z) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

Definition Let $s^{t+1} = (s^t, s_{t+1})$. The *one-step pricing kernel* is $Q_t(s_{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$. Note that, in a Markov environment, $q_t(s^t) \propto Q(s_t|s_{t-1})Q(s_{t-1}|s_{t-2})...Q(s_1|s_0)$.

Definition Suppose an asset pays dividends $d_k(s^k)$ in time k and state s^k, for $k = t, ..., \infty$ and s^k agrees with a certain history, s^t up to time t. This is called a *tail asset*. The price of this asset at time 0 contingent on history s^t occurring is $\sum_{k=t}^{\infty} \sum_{s^k|s^t} q_{t+k}^0(s^k|s^t) d_{t+k}(s^k|s^t)$. Note that $\frac{q_{t+k}^0}{q_{t+1}^0} = q_{t+k}^{t+1} = Q(s_{t+1}|s^t)$.

In general, we may price an asset from one period to the next as:

$$S_t = d_t(s^t) + \sum_{s \in S} q_{t+1}^t(s^t, s|s^t) S_{t+1}(s^t, s)$$

where (s^t, s) is the history at date $t+1$ that begins with s^t and ends at state s. This strips off the dividends before time t, looks only at the tail asset, and uses time t units of consumption instead of time 0 units of consumption (which scales everything by the constant, $\frac{1}{q_t^0(s^t)}$). (See if any of the derivation on the back of 42 is necessary; this is a more general result.)

Definition The *stochastic discount factor* is $\Lambda_t(s_{t+1}|s^t) = \frac{Q_t(s_{t+1}|s^t)}{\pi_t(s^t)} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$.

In equilibrium, the agent will try to equate the intertemporal merginal rate of substitution with the interest rate. That is, $1 + r = \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{1}{SDF}$. The SDF will change each period, depending on the realization.

Suppose we had complete markets at time 0 and then reopen trading at a later date. Then, endowments depend on both their incomes and their future claims (or future payments). That is, (financial) wealth at time τ is given by:

$$\Psi_\tau(\tilde{s}_\tau) = \sum_{t=\tau}^{\infty}\sum_{s^t|\tilde{s}^\tau} q_t^\tau(s^t)(c_t^i(s^t) - y_t^i(s_t))$$

($c_t^i(s^t) - y_t^i(s^t)$ are the net claims for that history). Prices at time τ are given by:

$$q_t^\tau(s^t) = \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \beta^{t-\tau}\frac{u'(c_t^i(s^t))}{u'(c_\tau^i(s^\tau))}\pi(s^t|s^\tau)$$

(see if the example on the back of 46 has anything else to add)

This means that there will be no trading if markets reopen.

In equilibrium, households choose their level of demand of c_t and their supply of k_{t+1}, n_t, while firms demand n_t, k_t and supply y_t. Governments choose g_t, and taxes. In equilibrium, by feasibility, everything must be equal.

Once we have computed a competitive equilibrium, it can be used as a benchmark for insurance, asset prices, government debt, and so on.

3.2 Arrow Securities

With Arrow securities, at time t, one can trade in markets based on the state at time $t+1$. In this case, there is trading in each period and fewer markets at each time. This sort of trading will require borrowing constraints.

For these economies, we assume that the states are a Markov chain, with initial density π_0 and transition density $\pi(s_t|s_{t-1})$, so that:

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_1|s_0)\pi_0(s_0)$$

We also assume that $y_t^i(s^t) = y^i(s_t)$, for stationarity.

We then have the Bellman equation for each household:

$$v(s,a) = \max_{c,a(s')}\left(u(c) + \beta\sum_{s\in S} v(s', a(s'))\pi(s'|s)\right)$$

subject to the budget constraint,

$$c + \sum_{s' \in S} Q(s', s)a(s') \leq y^i(s) + a$$

where $a(s)$ are the household assets in state s (also called the *Arrow securities*) and $Q(s', s)$ is the price of a claim on consumption tomorrow if today is state s and tomorrow is in state s'. Note that $a(s')$ may be negative. To prevent borrowing an infinite amount and consuming everything today (a Ponzi scheme), we also require that $c \geq 0$ and $-a^i(s') \leq A^i(s')$ for all i, s'; that is, the borrowing condition must hold across all states. Then, solving the household problem for each i will give the policy rule: $c = h(s, a(s))$, $a(s') = g(s', s, a(s))$.

We choose $A^i(s)$ and the pricing kernel to match the equilibrium from the Arrow-Debreu economy. (This is good because Arrow-Debreu is associated with the welfare theorems, but Arrow securities are more like reality; if one is efficient and they give identical answers, then the other must be as well.)

Definition The *natural borrowing limit* is given by $A(s) = \sum_{t=0}^{\infty} \sum_{s^t} q_0^t y_t^i(s^t)$. This is the present value of the future endowment, and ensures that we can always have $c \geq 0$ in future periods.

Definition The *j-step pricing kernel*, $Q_j(s'|s)$ is the price of consumption in j periods if today's state is s and the state j periods from now is s'. It can be computed recursively as $Q_1(s'|s) = Q(s'|s)$ and $Q_j(s_{t+j}|s_t) = \sum_{s_{t+1} \in S} Q_1(s_{t+1}|s_t)Q_{j-1}(s_{t+j}|s_{t+1})$.

In general, a recursive equilibrium is defined by:

- a value function,
- a policy function (which maps to demand),
- a pricing kernel, q, and
- borrowing limits and budget constraints.

3.3 Computing a Competitive Equilibrium

Generally, it is easier to compute an Arrow-Debreu equilibrium first and then match everything up to a Arrow securities equilibrium and work that out.

The welfare theorems must hold in the economy to ensure that the planner's solution and the equilibrium coincide.

An equilibrium is an allocation and a price system. In general, they are determined simultaneously. However, if there is a representative agent (this requires constant relative risk aversion with identical utilities and the same beliefs about probabilities), then the prices may be computed first. Under constant relative risk aversion,

$$c_t^i(s^t) = \phi_i \sum_{i=1}^{I} c_t^i(s^t) = \phi_i \sum_{i=1}^{I} y_t^i(s^t)$$

and there is complete risk sharing across all times and states. Then, prices are:

$$q_t^0(s^t) = \beta^t u' \left(\sum_{i=1}^{I} y_t^i(s^t) \right) \pi_t(s^t)$$

(because we may scale by a constant and $\frac{u'(c_1)}{u'(c_2)} = (\frac{c_1}{c_2})^{-\gamma}$). Any agent that does not have a corner solution can be used to compute prices (whether or not there are CRRA preferences), because all the ratios of marginal utilities are constant across consumers. Given these prices, we then solve the household problem to compute c_1 and then compute ϕ_1 from the budget constraints in the household problem.

To move from time 0 trading to sequential trading, we compute the Arrow securities prices as $Q(s_{t+1}|s_t) = \frac{q_{t+1}^0(s_{t+1}|s^t)}{q_t^0(s^t)}$.

In any complete market, there exists a representative consumer, with preferences equal to the indirect utility function of the planner over the aggregate endowment (which may depend on the Pareto weights). That is:

$$v(\bar{c}) = \max_{c^1,\ldots,c^I:\sum_{i=1}^{I} c_t^i(s^t)=\bar{c}(s^t)} \sum_{i=1}^{I} \theta_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u_i(c_t^i(s^t))\phi_t^i(s^t)$$

where the c^i, u_i, π^i may differ across the consumers, and the θ_i are Pareto weights. (With Gorman aggregation, the Pareto weights do not matter, because u, π_t are constant across consumers and the Engle curves are straight lines.)

4 Fiscal Policy and Growth

In this economy, we have:

- one (representative) individual with preferences given by $\sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t)$ where n_t is labor (and $1 - n_t$ is leisure),
- technology given by $c_t+x_t+g_t \leq F(k_t, n_t)$ and $k_{t+1} = (1-\delta)k_t+x_t$, where x_t is investment, F is a neo-classical production function with constant returns to scale, and δ is the depreciation rate, and
- a government with an exogenous stream of consumption, g_t, and a variety of exogenous taxes.

The household budget constraint is:

$$\sum_{t=0}^{\infty}(q_t(1+\tau_{ct})c_t+(1-\tau_{it})q_t(k_{t+1}-(1-\delta)k_t)) \leq \sum_{t=0}^{\infty}(r_t(1-\tau_{kt})k_t+w_t(1-\tau_{nt})n_t-q_t\tau_{ht})$$

where:

- τ_{ct} is a flat consumption tax,
- τ_{it} is an investment tax credit,
- τ_{kt} is a tax on rental income (capital gains),
- τ_{nt} is a tax on labor income,
- τ_{ht} is a lump sum ("head") tax,
- q_t is the time zero price of consumption (or government consumption or investment) at time t,
- r_t is the capital rental price, and
- w_t is the wage.

Consumers maximize utility, taking the price system, $(q_t, r_t, w_t)_{t=0}^{\infty}$, and taxes, $(\tau_t)_{t=0}^{\infty}$, as given.

The government budget constraint is:

$$\sum_{t=0}^{\infty} q_t g_t \leq \sum_{t=0}^{\infty} (\tau_{ct} c_t q_t - \tau_{it} q_t (k_{t+1} - (1-\delta) k_t) + \tau_{kt} k_t r_t + w_t \tau_{nt} n_t + q_t \tau_{ht})$$

A government policy, (g_t, τ_t), is *budget feasible* if it satisfies the budget constraint. Note that the private sector chooses c_t, k_t, n_t. Often, we assume that the head tax adjusts automatically to balance the budget.

A competitive equilibrium in this economy consists of a price vector, an allocation, and a policy vector such that the allocation solves the household's problem, taking prices and taxes as given, and the policy vector obeys the government's budget constraint.

We may rewrite the household's budget constraint to collect k_t terms:

$$\begin{aligned}\sum_{t=0}^{\infty} q_t c_t (1+\tau_{ct}) \quad \leq \quad & \sum_{t=0}^{\infty} w_t (1-\tau_{nt}) n_t - \sum_{t=0}^{\infty} q_t \tau_{ht} \\ & + \sum_{t=1}^{\infty} \left(r_t (1-\tau_{kt}) + q_t (1-\tau_{it})(1-\delta) - q_{t-1}(1-\tau_{i,t-1}) \right) k_t \\ & + (r_0 (1-\tau_{i0}) + (1-\tau_{i0}) q_0 (1-\delta)) - \lim_{T\to\infty} (1-\tau_{iT}) q_T k_{T+1}\end{aligned}$$

We must have the *no-arbitrage condition*:

$$r_t(1-\tau_{kt}) + q_t(1-\tau_{it})(1-\delta) - q_{t-1}(1-\tau_{i,t-1}) = 0$$

in every period. Otherwise, the consumer would want to amass an infinitely positive or negative amount of capital in that period, so they would have an infinitely high budget overall. We must have $\lim_{T\to\infty}(1-\tau_{iT}) q_T k_{T+1} = 0$ for the same reason.

The household first order conditions are:

$$\begin{aligned}\beta^t \frac{\partial U}{\partial c}(c_t, n_t) &= \mu q_t(1+\tau_{ct}) \\ \beta^t \frac{\partial U}{\partial n}(c_t, n_t) &\leq \mu w_t(1-\tau_{nt})\end{aligned}$$

We assume that F is linearly homogenous, so that, by Euler's Homogenous Function Theorem, $F(k,n) = F_k k + F_n n$, where F_k, F_n are the partial derivatives. Then, the value of the firm is $\sum_{t=0}^{\infty}(q_t F_k - r_t)k_t + (q_t F_n - w_t)n_t$. To ensure that this is bounded, we must have:

$$\begin{aligned}F_k q_t - r_t &= 0 \\ F_n q_t - w_t &= 0\end{aligned}$$

For simplicity, we assume that utility depends only on consumption, $U(c, 1-n) = u(c)$, with $u' > 0, u'' < 0, u'(0) = \infty$. Then, labor is supplied inelastically, and $n_t = 1$. This leaves a consumption first order condition:

$$\beta^t u'(c_t) = \mu q_t(1+\tau_{ct})$$

Because of constant returns to scale, we have $F(k,1) = f(k)$. Then, $F_k = f'(k)$ and $F_n = f(k) - f'(k)k$.

We can no longer use the welfare theorems to compute a competitive equilibrium since we have tax wedges. Instead, we use the first order conditions and the constraints directly:

$$\begin{aligned}k_{t+1} &= f(k_t) + (1-\delta)k_t - g_t - c_t \\ r_t &= q_t f'(k_t) \\ \beta^t u'(c_t) &= \mu q_t(1-\tau_{ct}) \\ u'(c_t) &= \beta u'(c_{t+1})\frac{1+\tau_{ct}}{1+\tau_{c,t+1}}\left(\frac{1+\tau_{i,t+1}}{1+\tau_{it}}(1-\delta) + \frac{1+\tau_{k,t+1}}{1+\tau_{it}}f'(k_{t+1})\right)\end{aligned}$$

The tax wedges distort the marginal choices; otherwise, we would have the usual Euler equations. We generally assume that $\{g_t, \tau_t\}_{y=0}^{\infty}$ are exogenous, and that the head tax, $\sum q_t \tau_{ht}$, automatically adjusts to satisfy the government budget constraint.

Given a sequence of spending and taxes, this gives us a system of difference equations with the initial condition, k_0, the boundary condition $\lim_{T\to\infty}(1-\tau_{it})\beta^T \frac{u'(c_T)}{1+\tau_{ct}} k_{T+1} = 0$, and forcing functions, g_t, τ_t that affect the choices of the control variables and therefore prices as well.

In an equilibrium, if $\overline{\tau}, \overline{g}$ are constant, then $\overline{q}, \overline{r}, \overline{w}, \overline{k}, \overline{c}, \overline{n}$ are all constant as well.

Let $z_t = (g_t, \tau_{it}, \tau_{kt}, \tau_{ct}, \tau_{nt})$. Suppose that these variables stabilize, so that $z_t = \bar{z}$ for all $t > \bar{T}$. This allows us to compute a steady state, if c_t and k_t converge as well. Substituting $k_t = k_{t+1} = \bar{k}$, we find that:

$$\begin{aligned}\bar{k} &= f(\bar{k}) + (1-\delta)\bar{k} - \bar{g} - \bar{c} \\ 1 &= \beta(1-\delta + f'(\bar{k})\frac{1-\bar{\tau}_k}{1-\bar{\tau}_i})\end{aligned}$$

We may solve the second equation for $\bar{k}$, and then solve the first equation for $\bar{c}$. If $\bar{\tau}_i = \bar{\tau}_k$ and we set $\beta = \frac{1}{1+\rho}$, then we find that $\rho + \delta = f'(\bar{k})$, which is the *augmented golden rule of capital*, which states that the marginal productivity of capital is the sum of the discount rate the the depreciation rate.

If taxes and capital do reach a steady state, then $\lim_{T\to\infty}(1-\tau_{iT})\beta^T \frac{u'(c_T)}{1+\tau_{cT}} k_{T+1} = 0$, since $\beta^T \to 0$ and everything else is constant.

Let R_{t+1} be the gross real interest rate, which is equal to the reciprocal of the price of consumption tomorrow. Then, in this economy,

$$R_{t+1} = \frac{1+\tau_{ct}}{1+\tau_{c,t+1}} \left(\frac{1-\tau_{i,t+1}}{1-\tau_{i,t}}(1-\delta) + \frac{1-\tau_{k,t+1}}{1-\tau_{it}} f'(k_{t+1}) \right)$$

Then we may rewrite the consumption Euler equation as $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. Under CRRA preferences, $u'(c) = c^{-\gamma}$, and we notice that:

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma}\log\beta + \frac{1}{\gamma}\log R_{t+1}$$

and high consumption growth is associated with high interest rates.

These can also be solved by using linear approximations and solving equations in terms of future variables or by using the Schur decomposition.

If consumption jumps in one period, then capital will jump in the next period.

4.1 Effects of Policies

Suppose the economy is at steady state, and then a policy adjustment is announced. We first compute the effect on the steady state. Then, we may use the shooting algorithm (or one of the other methods) to compute the path of the economy as it transitions.

In general, between the time a policy is announced and when it is implemented, people act differently because of *foresight*. After foresight has moved people away from the steady state, the economy moves back to the (possibly new) steady state in the *transient response*.

For example, suppose g_t increases permanently at time T. From time 0 (when the policy is announced) to time $T-1$, capital will increase, the interest rate will decrease, and consumption will decrease. After time T, capital will decline back to the steady state level (which is unchanged, since g does not affect the steady state), the interest rate will increase back to the old steady state, and consumption will continue declining to a new steady state (in which the additional government consumption has completely crowded it out).

If τ_{ct} has a foreseen increase at time T, there will be no change in steady state consumption, capital, or interest rate. However, consumption increases in anticipation of the increase in the consumption tax and then drops sharply when the tax is increased. Note that we can model a one-time increase in τ_{ct} as a one-time pulse in τ_{kt}, since an increase in the consumption tax leads to a one-time pulse in $\frac{1+\tau_{ct}}{1+\tau_{c,t+1}}$, which will affect the interest rate like a one-time pulse in $1 - \tau_{k,t+1}$. The consumption tax is not distorting when it is constant, but when it changes, it acts like a tax on capital, which is distorting.

4.2 Ricardian Equivalence

Suppose we have a representative agent with preferences $\sum_{t=0}^{\infty} \beta^t u(c_t)$, with $u'(c) > 0, u''(c) < 0, u'(0) = \infty$, so that $c_t \geq 0$ for all t. Assume $\{y_t\}$ is exogenous and known. We impose the budget constraint $c_t + R^{-1} b_{t+1} \leq y_t + b_t$, where b_t are the assets at time t. We assume that this is a small, open economy, so that $R > 1$ is a constant, known, risk-free interest rate for both borrowing and lending, fixed by the world economy.

There are two possible borrowing constraints:

- *No borrowing*: $b_t \geq \tilde{b}_{t+1} = 0$ for all t.
- *Natural borrowing constraint*: $b_{t+1} \geq \tilde{b}_{t+1} = -\sum_{j=0}^{\infty} R^{-j} y_{t+j}$, based on setting all future consumption to 0 in order to pay back the loan.

Households choose $(c_t, b_{t+1})_{t=0}^{\infty}$ to maximize their utility subject to $c_t + R^{-1} b_{t+1} \leq y_t + b_t$ and one of the borrowing constraints. Then, the first order conditions are $u'(c_t) \geq \beta R u'(c_{t+1})$ for all $t \geq 0$, where equality holds when borrowing is not constrained.

We assume that $\beta R = 1$. Then, if $b_{t+1} > 0$, $u'(c_t) = u'(c_{t+1})$ and $c_t = c_{t+1}$. That is, the consumers completely smooth consumption if their borrowing constraints allow. If $b_{t+1} = \tilde{b}_{t+1}$, then $u'(c_t) > u'(c_{t+1})$ and $c_t < c_{t+1}$. In this case, income must be higher in the future. That is, any corner solution means that the consumer is expecting income growth and would like to smooth consumption using future income.

At the period in which the present value of future income is maximized, the borrowing constraint no longer matters.

Suppose there are lump sum taxes, τ_t. Then, the consumer's new budget constraint and natural borrowing limit are:

$$
\begin{aligned}
c_t + R^{-1} b_{t+1} &\leq y_t + b_t - \tau_t \\
\tilde{b}_t &= \sum_{j=0}^{\infty} R^{-j} (\tau_{t+j} - y_{t+j})
\end{aligned}
$$

The government budget constraint is $B_t + g_t = \tau_t + R^{-1} B_{t+1}$, where B_t is government debt. The only constraint on government debt is that $\lim_{T \to \infty} R^{-T} B_{t+T} = 0$ for all t. We assume that the government can borrow even if households cannot.

In equilibrium, the government chooses $(g_t, \tau_t, B_{t+1})_{t=0}^{\infty}$ and the representative agent chooses $(c_t, b_{t+1})_{t=0}^{\infty}$ to satisfy their constraints and so that the household maximizes utility.

Proposition 4.1 The Ricardian Proposition. *Suppose that households are subject to the natural borrowing limit. Given initial condition,* (b_0, B_0)*, let* $\{\bar{c}_t, \bar{b}_{t+1}\}_{t=0}^{\infty}$,

$\{\overline{g}_t, \overline{\tau}_t, \overline{B}_{t+1}\}_{t=0}^{\infty}$ be an equilibrium. Consider any other tax policy, $\hat{\tau}_t$, that satisfies $\sum_{t=0}^{\infty} R^{-t}\hat{\tau}_t = \sum_{t=0}^{\infty} R^{-t}\overline{\tau}_t$. Then, $\{\overline{c}_t, \hat{b}_{t+1}\}_{t=0}^{\infty}$, $\{\overline{g}_t, \hat{\tau}_t, \hat{B}_{t+1}\}_{t=0}^{\infty}$ is an equilibrium, where $\hat{b} = \sum_{j=0}^{\infty} R^{-j}(\overline{c}_{t+j} - \hat{\tau}_{t+j} - y_{t+j})$ and $\hat{B}_t = \sum_{j=0}^{\infty} R^{-j}(\hat{\tau}_{t+j} - \overline{g}_{t+j})$.

Notice that the change in government borrowing and the change in private borrowing exactly offset each other, so that the private agents are just saving up to pay for future taxes. This proposition no longer holds in this form if agents are not allowed to borrow.

In the more general model with all forms of taxes, we may change the timing of the head tax without affecting anything, as long as the present value is unchanged, $\sum_{t=0}^{\infty} \overline{q}\tau_{ht} = \sum_{t=0}^{\infty} \overline{q}\tau_{ht}'$. This shows that deficits of a particular kind do not matter, because the head tax does not affect the constraints (because they only depend on the present value of the head tax), the no-arbitrage condition, or the first order conditions. This is a form of *Ricardian equivalence.*

Ricardian equivalence can be shown to hold in a sequential model as well. In that case, R^{-1} is the price of an Arrow security.

Ricardian equivalence applies to certain forms of distorting taxes as well.

4.2.1 Non-overlapping Generations Model

Suppose that instead of a single infinitely lived agent there is a sequence of agents that live one period. Assume each has the utility function, $V_t = u(c_t) + \beta V_{t+1}$, so that their total utility depends on their own consumption and on the total utility of their child. This can be written as $V_t = \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$. Each agent is subject to the budget constraints:

$$\begin{aligned} c_t + R^{-1}b_{t+1} &\leq y_t + b_t - \tau_t \\ b_t &\geq 0 \end{aligned}$$

where b_t is the bequest to the child.

To write this recursively, we assume that z_t is a finite-dimensional summary (and perfect predictor) of all future taxes and endowments. Then, $y_t = f(z_t)$, $\tau_t = h(z_t)$, and $z_{t+1} = g(z_t)$. Then, the state is (z_t, b_t), and we have the Bellman equation:

$$V(z_t, b_t) = \max_{c_t, b_{t+1}} u(c_t) + \beta V(z_{t+1}, b_{t+1})$$

with the constraints above.

This yields identical results to the previous model with no borrowing. As before, we can only get Ricardian equivalence if a new tax policy keeps all the bequests positive. In this case, this is called the *operational bequest motive.*

5 Rational Expectations Equilibrium

Definition A *rational expectations equilibrium* or *recursive competitive equilibrium* is a policy function, h, an actual aggregate law of motion, G_A, and a perceived law of motion, G, such that h solves the agent's problem given G and h implies that $G = G_A$. That is, when the agents take the perceived law of motion as given and optimize based on it, the perceived law of motion and the actual law of motion are identical.

The assumption of rational expectations ties down the agent's forcasting model to be the correct one.

To compute a rational expectations equilibrium in the linear case:

- Guess G_0, G_1 (the perceived law of motion).
- Compute h_0, h_1, h_2 (the choice of the agents given the perceived law of motion).
- Set $G_0 = h_0, G_1 = h_1 + h_2$ (the actual law of motion) and repeat.

This will not necessarily converge. This sets $G_j = T(G_{j-1})$.

We can also use a *learning algorithm*, in which we observe the history of p_t, Y_t and estimate G_0, G_1 from a regression on the history, optimizing based on those. This method is more likely to converge. This sets $G_j = \gamma T(G_{j-1}) + (1-\gamma)G_{j-1}$, for some $\gamma \in (0,1)$, which is more stable.

The rational expectations equilibrium is a fixed point in the perceived law of motion.

Some rational expectations equilibria can be transformed into a social planner's problem with the same first order conditions and solution.

Note that this differs from the standard Arrow-Debreu equilibrium. There, there is no uncertainty about prices or other endogenous variables. Rational expectations are needed because one must forecast future endogenous variables.

5.1 Example

Suppose we consider the partial equilibrium model in which a representative firm maximizes profits, where the price depends on industry output and changing the amount of output is costly. That is, the firm maximizes $\sum_{t=0}^{\infty} \beta^t R_t$ where

$$\begin{aligned} R_t &= p_t y_t - \frac{1}{2}(y_{t+1} - y_t)^2 \\ p_t &= A_0 + A_1 Y_t \\ Y_t &= n y_t \end{aligned}$$

If the firm is a price taker (which will necessarily happen if n is large), then the firm maximizes profits taking Y_t as given. This leads to a value function of:

$$v(y_0, Y_0) = \max_{y_1} (A_0 - A_1 Y_0) y_0 - \frac{d}{2}(y_1 - y_0)^2 + \beta v(y_1, Y_1)$$

subject to the law of motion $Y_1 = H(Y_0) = H_0 + H_1 Y_0$, where H_0, H_1 are the *perceived law of motion* by the firm of industry output. The firm's solution will lead to a linear decision rule of the form $y_{t+1} = h_0 + h_1 y_t + h_2 Y_t$, where h_0, h_1, h_2 depend on H, d, β, A_0, A_1. If $n = 1$, this leads to the *actual law of motion* which is $Y_{t+1} = h_0 + (h_1 + h_2)Y_t$.

In this case, the first order conditions for the firm are:

$$-d(y_{t+1} - y_t) + \beta(A_0 - A_1 Y_{t+1} + d(y_{t+2} - y_{t+1})) = 0$$

Since $Y_t = y_t$, this gives a second order difference equation in Y_t:

$$-d(Y_{t+1} - Y_t) + \beta(A_0 - A_1 Y_{t+1} + d(Y_{t+2} - Y_{t+1})) = 0$$

with boundary conditions that Y_0 is given and a limiting conditions. Then, this could be solved with the shooting algorithm or the Schur decomposition, which would yield $Y_{t+1} = H_0 + H_1 Y_t$. We can also use the difference equation:

$$dY_t - (d(1+\beta) + \beta A_1)Y_{t+1} + \beta d Y_{t+1} + \beta A_0 = 0$$

This difference equation must also come from a well-posed problem, and the solution will give the rational expectations equilibrium. In this case, we can integrate the demand curve to find the consumer surplus plus firm revenue:

$$\int_0^y (A_0 - A_1 s) ds = A_0 Y - \frac{1}{2} A_1 Y^2$$

The total social surplus is:

$$S_t = S(Y_t, Y_{t+1}) = \int_0^{Y_t} (A_0 - A_1 s) ds - \frac{1}{2} d(Y_{t+1} - Y_t)^2$$

This means that the firms' decision corresponds to the social planner's problem to maximize $\sum_{t=0}^{\infty} \beta^t S(Y_t, Y_{t+1})$, given Y_0, perhaps using the Bellman equation $V(Y_0) = \sum_{t=0}^{\infty} \beta^t S(Y_t, Y_{t+1})$. This will yield the same Euler equation and therefore is an equivalent problem.

6 Applications

6.1 The market for engineers

(Sherwin Rosen, 2004)

Consider the system of simultaneous equations:

$$\begin{aligned}
s_t &= a_0 + a_1 P_t + e_{st} \\
N_t &= (1-\delta)N_{t-1} + s_{t-k} \\
N_t &= d_0 - d_1 w_t + e_{dt} \\
P_t &= \sum_{j=0}^{\infty} \beta^{j+k} (1-\delta)^{j-k} E_t(w_{t+k+j}) \\
&= E_t(\beta(1-\delta)P_{t+1} + \beta^k (1-\delta)^k w_{t+k}) \\
\begin{pmatrix} e_{st} \\ e_{dt} \end{pmatrix} &\sim Normal(0, \begin{pmatrix} \sigma_s^2 & 0 \\ 0 & \sigma_d^2 \end{pmatrix})
\end{aligned}$$

where s_t is the supply of freshman engineers, k is the time to train an engineer, N_t is the stock of engineers, and w_t is the wage. In this model, the number of students entering depends on the expected present value of wages and an exogenous shock, the stock of engineers depreciates (as people retire) and new graduates enter, and the demand for engineers depends on the wage and an exogenous shock. A rational expectations equilibrium is a stochastic process, $\{s_t, N_t, w_t, P_t\}_{t=0}^{\infty}$, such that the equations hold simultaneously.

In this model, the state is $N_{-1}, s_{-1}, ..., s_{-k}$ while prices and wages are endogenous but not in the state.

Let y_t be a vector that includes both the state variables and the other endogenous variables. Then, we may write $Ly_{t+1} = Ny_t$ for some L, N. We may then invert L to find that $y_{t+1} = My_t$, and we may then find the Schur decomposition. Since such a solution will exist, this set-up must be related to a planning problem of the form $\sum_{t=0}^{\infty}(\beta(1-\delta))^t\Phi$, where the objective, Φ, depends on integrals of both s_t and N_t, which are the demand and supply curves. Then, these equations must be the first order conditions of some planning problem.

This model helps explain cycles in wages for jobs that require time for training.

6.2 Growth with Taxes

Based on Prescott's lecture, 2002.

In this model:

$$\begin{aligned} C_t + X_t &= Y_t = (Ae^{\gamma t})^{1-\theta} K_t^{\theta} H_t^{1-\theta} \\ K_{t+1} &= K_t - \delta K_t + X_t \end{aligned}$$

This model sets $e^{\gamma} = 1.02$ to match the long-run growth rate of GDP per capita from 1900 to the present and $\theta = 0.3$ to match the capital share of output in the US economy. The *stand-in household* chooses h_t, c_t to maximize $\sum_{t=0}^{\infty}\beta^t N_t(\log c_t + \alpha \log(1 - h_t))$, where α affects the labor supply elasticity. This model allows consumption to grow while hours worked has stayed relatively constant with economic growth. Households have the budget constraint:

$$\sum_{t=0}^{\infty} N_t p_t((1 + \tau_{ct})c_t + x_t - (1 - \tau_{nt})w_t h_t - r_t k_t + \tau_{kt}(r_t - \delta) - T_t) \leq 0$$

Note that all taxes are refunded as a lump sum and there is a depreciation allowance in the capital tax. Also, in this formulation, r_t, w_t are multiplied by p_t, the price level.

Using these equations, we find that, in per capita terms:

$$\begin{aligned} y_t &= \frac{Y_t}{N_t} \\ &= (Ae^{\gamma t})^{1-\theta} k_t^\theta h_t^{1-\theta} \\ \frac{1}{1-\theta}\log y_t &= \log A + \gamma t + \frac{\theta}{1-\theta}\log k_t + \log h_t \\ \log y_t &= \log A + \gamma t + \frac{\theta}{1-\theta}\log\left(\frac{k_t}{y_t}\right) + \log h_t \end{aligned}$$

This decomposes per capita GDP into a productivity factor, trend growth, a capital factor, and a labor factor.

This model yields the Lagrangian:

$$L = \sum_{t=0}^{\infty} N_t(\beta^t(\log c_t + \alpha\log(1-h_t)) + \mu p_t((1-\tau_{nt})w_t h_t + r_t k_t - \tau_{kt}(r_t-\delta)k_y + T - (1+\tau_{ct})c_t - k_{t+1} + (1-\delta)k_t))$$

Then, the no-arbitrage condition for capital is:

$$r_t p_t - p_t\tau_{kt}(r_t-\delta) - (1-\delta)p_t - p_{t-1} = 0$$

In the steady state, $p_t = \beta^t p_0$. Let $\beta = \frac{1}{1+i}$. Then:

$$\begin{aligned} 1+i &= \frac{1}{\beta} = \frac{p_{t-1}}{p_t} \\ &= r_t(1-\tau_{kt}) + \tau_{kt}\delta + (1-\delta) \\ r_t &= \frac{i}{1-\tau_{kt}} + \delta \end{aligned}$$

The first order conditions for the household are:

$$\begin{aligned} \frac{\beta_t}{c_t} &= \mu p_t(1+\tau_{ct}) \\ \alpha\beta^t\frac{1}{1-h_t} &= \mu p_t(1-\tau_{nt})w_t \end{aligned}$$

Dividing the two yields $1-h_t = \frac{1+\tau_{ct}}{1-\tau_{ht}} \cdot \frac{\alpha c_t}{w_t}$. Since the wage is the marginal product of labor, $w_t = (1-\theta)\frac{y_t}{h_t}$. This yields the equilibrium relationship:

$$h = \frac{1-\theta}{1-\theta+\frac{c}{y}\alpha(\frac{1+\tau_c}{1-\tau_h})}$$

This helps explain the difference in GDP per capita across countries using taxes, but ignores many other factors.

6.3 Labor Decisions

Suppose there are a large number of workers in the economy, with technology $y = f(n) = \gamma n$, so that γ is the real wage. The individual's utility is given by $U(c) - V(n)$, where U is concave and V is convex.

(Rogerson, Hansen.) Suppose $n \in \{0, 1\}$, so that the utility is $U(\gamma) - A$ if the individual works and $U(0)$ otherwise. An individual will choose to work if $U(\gamma) - A - U(0) > 0$; we assume a random utility model so that not all agents make the same decision. If $\frac{A}{U'(0)} < \gamma < \frac{A}{U'(\gamma)}$, the worker would choose to enter a lottery to determine whether he is employed. This might lead to state-contingent commodity trading so that the worker can equate the marginal utilities of consumption and leisure.

Suppose a fraction Φ of the population is chosen to work in the lottery. We require a planning problem to determines Φ. The planner maximizes $\Phi(u(c_1) - A) + (1 - \Phi)u(c_2)$ over Φ, c_1, c_2, subject to $\Phi c_1 + (1 - \Phi)c_2 = \Phi\gamma$, $c_1, c_2 \geq 0$, and $\Phi \in [0, 1]$. We then equate the fraction, Φ, to the ex ante probability. The optimal choice is $c_1 = c_2$. If A changes, then Φ changes as well.

If we include this decision in Prescott's utility function, we end up with an alternative equilibrium condition:

$$h = (1 + \frac{\alpha(c/y)}{1-\theta}\frac{1+\tau_c}{1-\tau_n}\frac{1}{1-\eta})^{-1}$$

where $\frac{1}{1-\eta}$ comes from the decision of whether to work.

6.4 Monetary Policy

(From Kyndland and Prescott)

Suppose there is a leader in the economy that wants to maximize $-\sum_{t=0}^{\infty}\beta^t(y_t'Ry_t + u_t'Qu_t)$, where $y_t = \begin{pmatrix} z_t \\ x_t \end{pmatrix}$, with z_t being the *natural state variables*, which are inherited from the past, and x_t being the *jump variables* which depend on both the past and future. (In monetary policy, the jump variables are the private sector's reaction to both policy and the economy as a whole.) The leader has a model for the economy:

$$\begin{pmatrix} I & 0 \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} z_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}\begin{pmatrix} z_t \\ x_t \end{pmatrix} + \hat{B}u_t$$

This includes both a law of motion for the state variables and a relationship for the jump variables (*implementability constraints* that depend on both the past and the future, based on the first order conditions of the private sector. If $\begin{pmatrix} I & 0 \\ G_{21} & G_{22} \end{pmatrix}$ is invertible, then we may write $y_{t+1} = Ay_t + Bu_t$. The leader's problem is to maximize subject to that law of motion.

This differs from a linear-quadratic optimization because x_0 is not given.

To solve this problem:

1. Ignore the fact that x_0 is not a state variable to compute a value function, $v(y) = -y'Py$, and policy, $u = -Fy$, as usual.
2. Recall that $\mu_t = Py_t$ (OR SOMETHING LIKE THAT), where the μ_t are the multipliers on the constraints. We can set μ_{xt}, the multiplier on the private sector constraints, to be a state variable (and choose it optimally in the first period).

Then, we can determine:

$$x_t = -P_{22}^{-1}P_{21}z_t + P_{22}^{-1}\mu_{xt}$$

This also yields a decision rule:

$$u_t = -F\begin{pmatrix} I & 0 \\ -P_{22}^{-1}P_{21} & P_{22}^{-1} \end{pmatrix}$$

and a law of motion:

$$\begin{pmatrix} z_{t+1} \\ \mu_{x,t+1} \end{pmatrix}\begin{pmatrix} I & 0 \\ P_{21} & P_{22} \end{pmatrix}(A - BF)\begin{pmatrix} I & 0 \\ -P_{22}^{-1}P_{21} & P_{22}^{-1} \end{pmatrix}\begin{pmatrix} z_t \\ \mu_{x,t} \end{pmatrix}$$

The optimal initial condition is $\mu_{x0} = 0$.

u_t depends on the entire history.

This assumes that the government chooses a single policy and sticks with it. μ_{xt} is the cost of maintaining that commitment. This cost increases over time. Sequential decisions would lead to different results.

6.5 Risk for the Long Run

Let Δc_t be consumption growth. We model it as:

$$\begin{aligned} \Delta c_t &= \mu + x_t + \epsilon_t^c \\ x_t &= \rho x_{t-1} + \epsilon_t^x \end{aligned}$$

We assume that $(\epsilon_t^c, \epsilon_t^x)$ are jointly normally distributed with covariance 0 and variances σ_c^2, σ_x^2.

Then, $\log C_t = \mu t + \sum_{j=0}^{t} x_j + \sum_{j=0}^{t} \epsilon_j^c$. The shocks to the trend are more persistent, which leads to more long-run risk (which is costly for risk-averse agents).

Based on quarterly data, we expect that the monthly autocorrelation in consumption growth is $\rho_1^m = 0.1$. On the other hand, based on asset pricing (the price/dividend ratio), we expect that $\rho = 0.979$. This means that $\frac{\sigma_x^2}{\sigma_c^2} \approx 0.0044$.

The price of an asset is given by:

$$P_t = E_t(\mu_{t+1}(D_{t+1} + P_{t+1}))$$

Dividends may grow over time, but the price-dividend ratio may be stationary:

$$\frac{P_t}{D_t} = E_t(\mu_{t+1}\frac{D_{t+1} + P_{t+1}}{D_t}) = E_t\sum_{j=1}^{\infty}\mu_{t+j}\frac{D_{t+j}}{D_t}$$