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Physics 111

A Simple Model of Ocean Tides

Ocean tides are caused by the gravitational attraction between the Earth and the moon and between the sun and the moon. There are, however, several difficulties in performing the calculation. To get around these, we will make several approximations. First of all, since the surface of the Earth is not an inertial system (due to its rotation), we will consider a separate inertial reference frame x', y', z'. We will also consider the Earth and the moon to be perfectly spherical, and we will consider the Earth to be covered completely by water. Let M_m be the mass of the moon, M_E be the mass of the Earth, r be the radius of the Earth, and D be the distance from the center of the moon to the center of the earth. Now consider a small mass m on the surface of the earth. Let $\mathbf{R}$ be the vector distance from the center of the moon to m, $\mathbf{r}$ be the distance from the center of the Earth, and $\mathbf{r}'_m$ be the distance from the origin of the inertial reference frame. Finally, let $\mathbf{r}'_E$ be the distance from the inertial system to the center of the Earth. We can now write down the total force on m due to the Earth and the moon:

$$m\ddot{r}'_m = -\frac{GmM_E}{r^2}\mathbf{e}_r - \frac{GmM_m}{R^2}\mathbf{e}_R$$

We can also find the force exerted on the center of the Earth by the moon:

$$M_E\ddot{r}'_E = -\frac{GM_E M_m}{D^2}\mathbf{e}_D$$

Using these two equations, we can determine the total acceleration of m in the non-inertial frame at the center of the Earth. This total acceleration $\ddot{r}$ is simply the difference of the two accelerations determined above:

$$\begin{aligned}\ddot{r} &= \ddot{r}'_m - \ddot{r}'_E == \frac{GM_E}{r^2}\mathbf{e}_r - \frac{GM_m}{R^2}\mathbf{e}_R + \frac{GM_m}{D^2}\mathbf{e}_D \\ &= -\frac{GM_E}{r^2}\mathbf{e}_r - GM_m\left(\frac{\mathbf{e}_R}{R^2} - \frac{\mathbf{e}_D}{D^2}\right)\end{aligned}$$

We can see that the first part of this acceleration is just g, the normal acceleration due to gravity. Therefore, it must be the second part of the acceleration that is due to tidal forces. Note that it depends on the difference between the strength of the moon's gravity at the surface of the Earth and at the center. We can see, then, that tidal force can be expressed as

$$\mathbf{F}_T = -GmM_m\left(\frac{\mathbf{e}_R}{R^2} - \frac{\mathbf{e}_D}{D^2}\right)$$

Let us first look at this force when both $\mathbf{e}_R$ and $\mathbf{e}_D$ lie along the same line (for convenience, we'll call this the x-axis). Noting that $R = D + r$, we have

$$F_{T_x} = -GmM_m\left(\frac{1}{R^2} - \frac{1}{D^2}\right) = -GmM_m\left(\frac{1}{(D+r)^2} - \frac{1}{D^2}\right) = -\frac{GmM_m}{D^2}\left(\frac{1}{\left(1+\frac{r}{D}\right)^2} - 1\right)$$

Expanding $\left(1+\frac{r}{D}\right)^{-2}$ and noting that $\frac{r}{D}$ is very small (≈ 0.02) we have

$$F_{T_x} = -\frac{GmM_m}{D^2}\left[1 - 2\frac{r}{D} + 3\frac{r^2}{D^2} - \cdots - 1\right] \approx \frac{2GmM_m r}{D^3}$$

Now, we would like to find the force along the y-axis, and so we consider the tidal force at a point along the surface of the Earth displaced 90 degrees from the x-axis. At this point, since $R \approx D$ and the horizontal components of $\mathbf{e}_R$ and $\mathbf{e}_D$ will be approximately equal, we can make the approximation that the horizontal component of the tidal force will be zero. The vertical component of $\mathbf{e}_D$ will be zero since D is completely horizontal, and so we are left simply with the vertical component of $\mathbf{e}_R$. Using a small angle approximation, we can say that this vertical component will be $\frac{r}{D}\hat{\mathbf{j}}$, and so the vertical tidal force will be

$$F_{T_y} = -GmM_m\left(\frac{1}{D^2}\frac{r}{D}\right) = -\frac{GmM_m r}{D^3}$$

We can now find the tidal force on an arbitrary point along the surface of the Earth by substituting $x = r\cos\theta$ and $y = r\sin\theta$ for r in our equations for F_{T_x} and F_{T_y}, respectively. We then find that the tidal force on an arbitrary point along the surface of the Earth is

$$\mathbf{F}_T = \frac{GmM_m r}{D^3}\left(2\cos\theta\hat{\mathbf{e}}_x - \sin\theta\hat{\mathbf{e}}_y\right)$$

An identical calculation for the effect of the sun on the Earth, using R_{Es} as the distance from the center of the sun to the Earth and M_s as the mass of the sun leads to an expression for the tidal force due to the sun:

$$\mathbf{F}_T = \frac{GmM_s}{R_{Es}^3}\left(2\cos\theta\hat{\mathbf{e}}_x - \sin\theta\hat{\mathbf{e}}_y\right)$$

Following a method of Newton's, we can now calculate an approximate value for the maximum tidal height due to both the moon and the sun. Imagine that we can dig two wells to the center of the Earth, one along the low tide direction and one along the high tide direction. If we now move a mass from the top of one well through the center of the earth to the top of the other well, gravity will do work on it, and this work will equal the change in gravitational potential energy. Since potential energy can be expressed as mgh, we can use this to find the maximum tidal height. The work done (again assuming a spherical Earth) is

$$W_m = \int_r^0 F_{T_y}dy + \int_0^r F_{T_x}dx = \frac{GmM_m}{D^3}\left[\int_r^0 (-y)dy + \int_0^r 2xdx\right] = \frac{3GmM_m r^2}{2D^3}$$

for the tidal force due to the moon and

$$W_s = \frac{3GmM_s r^2}{2R_{Es}^3}$$

for the tidal force due to the sun. The ratio of these values is (Problem 5-18)

$$\frac{M_m}{M_s}\left(\frac{R_{Es}}{D}\right)^3$$

We have made many approximations to obtain these values, and so they are not quite accurate. First of all, we have neglected ocean depth and the complex effects of shorlines and continental shelfs. Even though it turned out not to matter what object we were calculating the force on (which leads to the interesting result that even land undergoes tides), there can exist resoncances that will influence tidal changes. Our results, though, are fairly accurate for the middle of the ocean. Additionally, both the moon and the Earth rotate. This is the reason why there are not exactly two tides per day. In fact, we can calculate the frequency of high tide given the periods of rotation of both the Earth and the moon (Problem 5-19). The Earth rotates about its axis once every 24 hours, and the moon revolves around the Earth in the same direction as the Earth's spin every 27.3 days. There will be a high tide at a particular point on the Earth when the moon is directly above it or directly below it. So, if we subtract the Earth's angular velocity from the moon's and thereby create a system where the Earth is stationary and the moon moves around it, we have only to calculate the new period of the moon and divide it by two to find out how frequently a given spot will experience high tide.

$$\frac{1}{2}\tau_T = \frac{1}{2}\frac{2\pi}{\frac{2\pi}{\tau_E} - \frac{2\pi}{\tau_M}} = \frac{1}{2}\frac{\tau_M\tau_E}{\tau_M - \tau_E} = 12\ hours, 26\ minutes$$