Extra Problem [32]

(a) Here we have an electron in the right half-space with a potential function \( V(x) = -\frac{e^2}{4x} \). For large \( x \), \( V(x) \to 0 \), and so we must have
\[
\frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi = -|E| \psi,
\]
which is solved by
\[
\psi(x) = e^{-\alpha x}, \quad \alpha = \frac{2m}{\hbar} |E|,
\]
since the positive exponential solution blows up as \( x \to \infty \), and is therefore unphysical.

(b) The boundary condition on \( \psi(x) \) at \( x = 0 \) is \( \psi(0) = 0 \).

(c) Let \( \psi(x) = f(x)e^{-\alpha x} \). Then we have
\[
\frac{d\psi}{dx} = -\alpha f e^{-\alpha x} + f' e^{-\alpha x}
\]
and
\[
\frac{d^2 \psi}{dx^2} = e^{-\alpha x} f'' - 2\alpha e^{-\alpha x} f' + \alpha^2 e^{-\alpha x} f.
\]
Plugging this back into the Schrödinger equation, we have
\[
f'' - 2\alpha f' + \frac{2me^2}{4\alpha^2} f + \frac{2m|E|}{\hbar^2} f = \frac{2m|E|}{\hbar^2} f.
\]
This simplest function of \( x \) that will solve this equation is \( f(x) = x \), and we know that the solution of this equation is unique. Putting this into equation (2), we find
\[
\alpha = \frac{me}{4\hbar^2} = \frac{1}{4a_0},
\]
where \( a_0 \) is the Bohr radius.

All that remains now is to normalize \( \psi(x) = Nxe^{-\alpha x} \), where \( N \) is a normalization constant. Integrating the absolute square of \( \psi \) and making the substitution \( y = 2\alpha x \), we have
\[
1 = N^2 \int_0^\infty x^2 e^{-2\alpha x} dx = \frac{N^2}{8\alpha^3} \int_0^\infty y^2 e^{-y} dy = \frac{N^2}{8\alpha^3} \Gamma(3),
\]
and thus that \( N^2 = 4\alpha^3 \). Therefore,
\[
\psi(x) = \sqrt{4\alpha^3} xe^{-\alpha x}
\]
in the ground state.

(d) Using equations (1) and (3), we easily find the ground state energy:
\[
|E| = \frac{\hbar^2}{2m} \alpha^2 = \frac{\hbar^2 a_0^2}{2m} \left( \frac{me^2}{\hbar^2} \right)^2 = \frac{me^4}{32\hbar^2}.
\]
(e) The expectation value of the position operator in the ground state is

\[
\langle \psi | \hat{x} | \psi \rangle = \int_0^\infty \int_0^\infty \langle \psi | x' \rangle \langle x' | \hat{x} | x \rangle \langle x | \psi \rangle dx dx' = \int_0^\infty \int_0^\infty x \langle \psi | x' \rangle \langle x | \psi \rangle \delta(x - x') dx dx' \\
= \int_0^\infty x \langle \psi | x \rangle \langle x | \psi \rangle dx = \int_0^\infty x \psi^* \psi dx = N^2 \int_0^\infty x^3 e^{-2\alpha x} dx = \frac{4\alpha^3}{(2\alpha)^3} \Gamma(4) \\
= \frac{6\hbar^2}{me^2} = 6a_0.
\]