Extra Problem [99]

We are given the potential

\[ V(x) = \begin{cases} 
  -\frac{K}{x}, & x > 0 \\
  \infty, & x \leq 0
\end{cases} \]

In extra problem 32, we found that the ground state energy was

\[ \epsilon = \frac{mK}{2\hbar^2} \]

and that the ground state wavefunction was

\[ \psi(x) = 2\left(2\epsilon\right)^{3/2}xe^{-2\epsilon x}. \]

Let us now apply an electric field \( \vec{E} = E_0 \hat{x} \). We can compute the Stark effect shift to the ground state energy, noting that \( \hat{H}_{St} = -e\hat{x}E_0 \). To first order, the new ground state energy will be

\[ E = \epsilon + \langle 1 | \hat{H}_{St} | 1 \rangle = \epsilon - eE_0 \langle 1 | \hat{x} | 1 \rangle. \]

Computing this matrix element, we have

\[ \langle 1 | \hat{x} | 1 \rangle = \int_0^\infty x\psi^*\psi dx = 4(2\epsilon)^3 \int_0^\infty x^3e^{-4\epsilon x} dx = 4(2\epsilon)^3 \frac{\Gamma(4)}{(4\epsilon)^4} = \frac{3}{4\epsilon}. \]

Thus, the ground state energy will be

\[ E = \epsilon - \frac{3eE_0}{4\epsilon} = \frac{mK^2}{2\hbar^2} - \frac{3\hbar^2 eE_0}{2mK^2}, \]

to first order.

Extra Problem [100]

(a) The ground state wavefunction will be identical to that of hydrogen, if we replace all the \( a_0 \)’s by \( 2a_0 \). This occurs since \( a_0 \propto 1/\mu \), and in this case the reduced mass is \( m/2 \). Thus, the ground state wavefunction is

\[ \psi_{100}(r) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{2a_0} \right)^{3/2} e^{-r/2a_0}. \]

(b) To find the root mean square radius for this state, let us first find the average square radius:

\[ \langle r^2 \rangle = \frac{1}{\pi} \left( \frac{1}{2a_0} \right)^3 \int_0^\infty 4\pi r^4 e^{-r/a_0} = \frac{1}{2a_0^3} a_0^5 \Gamma(5) = 10a_0^2. \]

Thus, the root mean squared radius is

\[ \langle r^2 \rangle^{1/2} = \sqrt{10}a_0, \]

and this is an estimate of the diameter of positronium.

(c) We are now given a hyperfine interaction Hamiltonian of

\[ \hat{H}_{int} = -\frac{8\pi}{3} \mu_e \cdot \mu_p \delta(r), \]

where

\[ \mu = \frac{ge}{2mc} \hat{S}. \]
The ground state energy of this perturbed system will be

\[ E = \epsilon + (100ss_z|\hat{H}_{\text{int}}|100ss_z). \]

We can separate this matrix element into a radial part and a spin part and evaluate each separately. Let us first evaluate the radial part:

\[ -\frac{8\pi}{3} \langle 100\delta(r)|100 \rangle = \left( \frac{8}{3} \right) \left( \frac{1}{2a_0} \right)^3 \int_0^\infty e^{-r/a_0}\delta(r)dr d\Omega = -\frac{1}{3a_0^3}. \]

To evaluate the spin part, we first use the identity

\[ \hat{S}_e \cdot \hat{S}_p = \frac{1}{2} \left( \hat{S}_e^2 - \hat{S}_p^2 \right). \]

We then have

\[ \left( \frac{ge}{2mc} \right)^2 \langle s's'_z|\hat{S}_e^2 - \hat{S}_p^2|ss_z \rangle = \left( \frac{ge}{2mc} \right)^2 \left( s's'_z \right) \frac{1}{2} \left( \hat{S}_e^2 - \frac{3}{2} \hat{h}^2 \hat{I} \right) \delta_{s's'_z}. \]

In the singlet state, \( s = 0 \), and so this matrix element will be

\[ -\frac{3\hbar^2}{4} \left( \frac{ge}{2mc} \right)^2, \]

and in the triplet state, \( s = 1 \), and so the matrix element becomes

\[ \frac{\hbar^2}{4} \left( \frac{ge}{2mc} \right)^2. \]

Our perturbed energy levels are then

\[ E_S = \epsilon + \frac{\hbar^2}{4a_0^3} \left( \frac{ge}{2mc} \right)^2 \]

for the singlet state and

\[ E_T = \epsilon - \frac{\hbar^2}{12a_0^3} \left( \frac{ge}{2mc} \right)^2 \]

for the triplet state. The energy splitting, then, is

\[ \frac{\hbar^2}{3a_0^3} \left( \frac{ge}{2mc} \right)^2, \]

which can be reexpressed as

\[ \frac{1}{3a_0} \left( \frac{e^2}{\hbar c} \right)^2, \]

which is a third of twice the ground state energy of hydrogen times the fine structure constant squared. Numerically, the energy splitting is

\[ \frac{2}{3}(13.6)\alpha^2 = 4.83 \times 10^{-4}\text{eV} = 117\text{GHz}. \]