**Extra Problem [86]**

(a) Here we have a cylindrically symmetric system governed by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + V(\hat{\rho}), \quad \hat{\rho} = \sqrt{x^2 + y^2}.$$  

Since this force is central in the plane, the $z$-component of the angular momentum must be a constant of the motion, and therefore $[\hat{H}, \hat{L}_z] = 0$. Since the potential does not depend on $z$, the $z$-component of the linear momentum will also be a constant, and so $[\hat{H}, \hat{p}_z] = 0$.

(b) Since $\hat{H}$, $\hat{L}_z$, and $\hat{p}_z$ share a common eigenbasis, when the separation of variables method is applied to the Schrödinger equation, we will find solutions of the form

$$\psi(x) = R(\rho)e^{im\phi}e^{ikz},$$

where $e^{im\phi}$ are the eigenfunctions of $\hat{L}_z$ and $e^{ikz}$ are the eigenfunctions of $\hat{p}_z$.

(c) In cylindrical coordinates, Schrödinger’s equation becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{2\mu}{\hbar^2} V \psi = -\frac{2\mu E}{\hbar^2} \psi.$$  

Substituting in our SOV solution, we have

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2} (-m^2 + (-k^2) - 2\mu \hbar^2 V) R = -\lambda^2 R,$$

where $\lambda^2 = \frac{2\mu E}{\hbar^2}$. After some simplification, this becomes

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + \left( \rho^2 \lambda^2 - (m^2 + k^2) \right) R - \frac{2\mu}{\hbar^2} \rho^2 V R = 0.$$

(d) Now we set

$$V(\rho) = \begin{cases} 0 & \rho < a \\ \infty & \rho > a \end{cases}$$

The wavefunction goes to zero for $\rho > a$, and for $\rho < a$ the radial equation becomes Bessel’s equation. Letting $\alpha^2 = m^2 + k^2$, the solution to the radial equation is

$$R(\rho) = AJ_\alpha(\lambda \rho),$$

where we have excluded the Bessel functions of the second kind since they are not normalizable at the origin. The energy eigenvalues are then given by

$$E_{n\alpha} = \frac{\hbar^2}{2\mu a^2} z_{n\alpha}^2,$$

where $z_{n\alpha}$ is the $n^{th}$ zero of $J_\alpha$. The three lowest energy eigenvalues for states with $\hat{p}_z$ and $\hat{L}_z$ equal to zero are then specified by the first three zeros of $J_0$, and are

$$E_1 = \frac{\hbar^2}{2\mu a^2} (2.4048)^2, \quad E_2 = \frac{\hbar^2}{2\mu a^2} (5.5201)^2, \quad E_3 = \frac{\hbar^2}{2\mu a^2} (8.6537)^2.$$
(e) For states with a nonzero $\hat{L}_z$, the lowest three energy eigenvalues are given by the first zeros of $J_0$, $J_1$, and $J_2$, and are

$$E_1 = \frac{\hbar^2}{2\mu a}(2.4048)^2, \quad E_1 = \frac{\hbar^2}{2\mu a}(3.8317)^2, \quad E_1 = \frac{\hbar^2}{2\mu a}(5.1356)^2.$$