Extra Problem [71]

Here we have a particle of mass \( m \) moving in the potential

\[
V(r, \theta, \phi) = \begin{cases} 
\infty, & b < r \quad \text{region I} \\
0, & a < r < b \quad \text{region II} \\
\infty, & r < a \quad \text{region III}
\end{cases}
\]

The particle is not permitted in regions I or III, so

\[
\psi_I(r, \theta, \phi) = \psi_{III}(r, \theta, \phi) = 0.
\]

In region II, we can write down the Schrödinger equation in spherical coordinates, recognizing that the angular part of the equation is simply the \( L^2 \) operator:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{L^2 \psi}{\hbar^2 r^2} = -\frac{2mE}{\hbar^2} \psi.
\]

We then have that

\[
\psi(r, \theta, \phi) = R_\ell(r) Y_m^\ell(\theta, \phi),
\]

and this gives us a radial equation of

\[
\frac{d^2 R_\ell}{dr^2} + \frac{2}{r} \frac{dR_\ell}{dr} - \frac{\ell(\ell + 1)}{r^2} R_\ell + k^2 R_\ell = 0,
\]

where

\[
E = \frac{k^2 \hbar^2}{2m}.
\]

This radial equation is Bessel’s equation in spherical coordinates, and so the solution is

\[
R_\ell(r) = A j_\ell(kr) + B \eta_\ell(kr).
\]

Since \( r = 0 \) is not in this problem, we cannot throw out the \( \eta_\ell \) solutions. At this point, finding the general eigenfunction is difficult, so we will find only the ground state eigenfunction and energy. The first spherical Bessel functions are

\[
j_0(x) = \frac{\sin x}{x} \quad \text{and} \quad \eta_0(x) = -\frac{\cos x}{x},
\]

and thus the radial part of the ground state wavefunction will be

\[
R_0(r) = A \frac{\sin kr}{r} - B \frac{\cos kr}{r}.
\]

Now we impose boundary conditions. We know that since we have an infinite potential, the wavefunction must be zero at both \( r = a \) and \( r = b \). This means that

\[
A \sin ka = B \cos ka
\]

and

\[
A \sin kb = B \cos kb.
\]
Thus, we must have
\[ \tan ka = \tan kb, \]
which in turn means that
\[ \sin [k(b - a)] = 0. \]
Thus, we see that
\[ k = k_n = \frac{n\pi}{b - a}, \]
which gives us a ground state energy of
\[ E_1 = \frac{\hbar^2 \pi^2}{2m(b - a)}. \]
Using the fact that
\[ Y_0^0 = \frac{1}{\sqrt{4\pi}}, \]
the ground state wavefunction is
\[ \psi_g(r, \theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{1}{r} (A \sin k_1 r - B \cos k_1 r). \]
Using the fact that
\[ \frac{B}{A} = \tan k_1 a, \]
this wavefunction can be rewritten as
\[ \psi_g(r, \theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{A}{r} (\sin k_1 r - \tan k_1 a \cos k_1 r) = \frac{1}{\sqrt{4\pi}} \frac{A}{r} \sin \left[ \frac{\pi(r - a)}{b - a} \right]. \]
For this wavefunction to be normalized, we must have
\[
1 = A^2 \int_a^b \sin^2 \left[ \frac{\pi(r - a)}{b - a} \right] dr.
\]
Using \( \rho = r - a \), we have
\[
1 = A^2 \left( \frac{\rho}{2} - \frac{\sin \left( \frac{2\pi \rho}{b - a} \right)}{\frac{2\pi}{b - a}} \right) \bigg|_0^{b-a} = A^2 \left( \frac{b - a}{2} \right). \]
Thus,
\[ A = \sqrt{\frac{2}{b - a}}, \]
and the normalized ground state wavefunction is
\[ \psi_g(r, \theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{1}{r} \sqrt{\frac{2}{b - a}} \sin \left[ \frac{\pi(r - a)}{b - a} \right]. \]