Nick Ouellette
Physics 113

Extra Problem [73]

(a) Here we have a particle moving in a potential

\[ V(r) = C \ln \left( \frac{r}{r_0} \right), \]

and we would like to know about its mean-squared velocity. We can change this into asking about something we do know about in the following way:

\[ \langle v^2 \rangle = \langle \frac{p^2}{m^2} \rangle = \frac{2}{m} \langle T \rangle. \]

To get at this expectation value, we can use the virial theorem. Let us define a quantity \( S \) such that

\[ S \equiv p \cdot r. \]

Its time derivative is

\[ \dot{S} = p \cdot \dot{r} + \dot{p} \cdot r, \]

and the expectation value of this time derivative is

\[ \langle \dot{S} \rangle = \frac{1}{\tau} \int_0^\tau \dot{S} dt = \frac{S(\tau) - S(0)}{\tau}. \]

As \( \tau \) gets large, this time average will go to 0, and so we can say

\[ \langle p \cdot \dot{r} \rangle = -\langle \dot{p} \cdot r \rangle. \]

The left side of this equation is simply twice the kinetic energy of the system, and using Newton’s second law we can rewrite the right hand side in terms of the force:

\[ \langle 2T \rangle = -\langle F \cdot r \rangle. \]

Using the fact that \( F = -\nabla V \), we have

\[ \langle 2T \rangle = \langle r \cdot \nabla V \rangle, \]

which is the general form of the virial theorem. We have now expressed the mean-squared velocity in terms of the potential, which we know. For us,

\[ \nabla V = \frac{C}{r}, \]

and so

\[ \frac{1}{m} \langle 2T \rangle = \int_{0}^{\infty} \frac{r \cdot \nabla V}{m} |\psi|^2 d^3r = \frac{C}{m} \int_{0}^{\infty} |\psi|^2 d^3r = \frac{C}{m}. \]

Thus, \( \langle v^2 \rangle = \frac{C}{m} \), and so clearly the mean-squared velocity is independent of state.
(b) To examine how the spacing between energy levels varies with the mass, we first look at how an arbitrary energy level varies with \( m \). We can express this in terms of the Hamiltonian as follows:

\[
\frac{\partial E_n}{\partial m} = \left\langle \frac{\partial \hat{H}}{\partial m} \right\rangle = \left\langle \frac{\partial}{\partial m} \left[ \frac{\hat{p}^2}{2m} + V(r) \right] \right\rangle.
\]

The potential is independent of mass, so this derivative reduces to

\[
\frac{\partial E_n}{\partial m} = \left\langle -\hat{p}^2 \cdot \frac{1}{2m^2} \right\rangle = \left\langle \frac{1}{2} \hat{v}^2 \right\rangle = \frac{C}{2m},
\]

where we have used the result from part (a). Now, we can look at how the energy level spacing varies with \( m \):

\[
\frac{\partial}{\partial m} (E_n - E_{n'}) = \frac{C}{2m} - \frac{C}{2m} = 0.
\]

Thus, the spacing between the energy levels is independent of the mass.