Renormalization Group Theory for the Ising Model

Renormalization group theory exploits the observation that at a critical point, a thermodynamic system exhibits self-similarity on all length scales. If we can solve the problem of one length scale, we can solve them all. The theory uses an iterative method to solve the problem, as will be shown below. Here, we focus only on the Ising model, but renormalization group theory has much broader application.

Let us begin with the 1-dimensional case. Here, there will only be interactions between nearest neighbor spins, and so we can write the partition function as

\[ Z = \sum e^{\beta J(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \cdots)}, \]

where we are summing over \( \pm 1 \) for each spin. Let us define \( K \equiv \beta J \), and let us also rewrite the partition function as

\[ Z = \sum e^{K(\sigma_1 \sigma_2 + \sigma_2 \sigma_3)} e^{K(\sigma_3 \sigma_4 + \sigma_4 \sigma_5)} \cdots = 2^{N/2} \cosh[K(\sigma_1 + \sigma_3)] \cosh[K(\sigma_3 + \sigma_5)] \cdots, \]

where the last equality follows from summing over all the even spins. Our goal now is to rewrite this form of the partition function in a form equivalent to our original partition function so that we can iterate our procedure. Expressed mathematically, we want a function \( f \) and a constant \( K' \) such that

\[ 2 \cosh[K(\sigma_i + \sigma_{i+2})] = f(K) e^{K' \sigma_i \sigma_{i+2}}. \]

Recognizing that the \( \sigma \)'s can be only \( \pm 1 \), we can write this as two equations for two unknowns, and the solution of the system is

\[ f(K) = 2(\cosh 2K)^{1/2}, \]
\[ K' = \frac{1}{2} \ln(\cosh 2K). \]

We can now rewrite the partition function as

\[ Z(N, K) = f(K)^{N/2} \sum e^{K' \sigma_1 \sigma_3} e^{K' \sigma_3 \sigma_5} \cdots = f(K)^{N/2} Z\left(\frac{N}{2}, K'\right). \]

This is not quite in a form we can iterate yet, though, since the two partition functions define systems with different numbers of particles. We can use the fact that the logarithm of the partition function is extensive, though, to write

\[ \ln Z(N, K) = N \zeta(K), \]

where \( \zeta \) is equivalent to the partition function with any extensive variable dependence removed. Thus, taking the logarithm of the recursively defined partition function above, we have

\[ \ln Z(N, K) = \ln \left[ f(K)^{N/2} Z\left(\frac{N}{2}, K'\right)\right], \]

or

\[ \zeta(K') = 2\zeta(K) - \frac{1}{2} \ln(\cosh 2K) - \ln 2. \]

We now have a recursion relation for the partition function we want to calculate in terms of a system with a doubled length scale (since the new function describes a system with only the odd spins from our original
system). Using iteration, then, we can start with the largest system, which is dilute enough that we can ignore its interactions, and progressively calculate down to the smallest length scale, which is our desired result. To perform these calculations, it is easier to use the inverses of the recursion relations derived above, which are shown in Baierlein.

The two dimensional case is both more interesting and more difficult. It undergoes a phase transition at some non-trivial critical point, but also involves more than nearest neighbor interactions. A sketch of the derivation of the recursion relations is shown here. Consider a two dimensional square lattice:

\[
\begin{array}{ccccccc}
  & & & & & & \\
  & & & & & & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  & \bigcirc & \bigcirc_1 & \bigcirc & \vdots & \vdots & \\
  & \bigcirc_2 & \bigcirc_5 & \bigcirc_4 & \vdots & \vdots & \\
  & \bigcirc & \bigcirc_3 & \bigcirc & \vdots & \vdots & \\
  & & & & & & \\
\end{array}
\]

The partition function is a straightforward extension of the one dimensional partition function. Let us average (sum) over spin 5 in the picture above, as well as all other center spins (e.g., every fifth spin) We arrive at a partition function of the form

\[Z(N, K) = \sum 2^{N/2} \cosh(K(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4))\]

We cannot reduce this expression using only a single new coupling constant as we did in the one dimensional case, since each term in this partition function has four degrees of freedom. Instead, we will need three new coupling constants, which will couple not only nearest neighbors, but also next-nearest neighbors and the four corners of each square. These constants can be calculated (see Chandler), but are difficult to deal with. We can approximate the solution by first ignoring the four corner interactions, since that coupling constant is small. If we also ignored the next-nearest neighbor interactions, we would have a simple problem, but we would have reduced it to the one dimensional case and would not find a phase transition. Instead, we must combine the two remain coupling constants together in such a way that we can reduce the problem to something we can solve but retain the phase transition. Using these approximations, we arrive at a recursion relation of the form

\[\zeta(K) = \frac{1}{2} \ln f(K) + \frac{1}{2} \ln f(K'),\]

where

\[K' = \frac{3}{8} \ln(\cosh 4K)\]

and

\[f(K) = 2^{[\cosh 2K]^{1/2}} [\cosh 4K]^{1/8}.\]