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Physics 112

The Tensor Formulation of Relativistic Electrodynamics

Einstein's theory of Special Relativity grew out of classical electrodynamics. We have seen that applying the theory to classical mechanics necessitates a rethinking of the relationship between the fundamental quantities in mechanics. When applied to electrodynamics, however, relativity provides a firm theoretical basis for the phenomenology of magnetism and its relation to electricity (shoring up the belief that the two are fundamentally related), and even goes so far as to explain why we observe magnetism at all. As an added bonus, relativity allows us to express Maxwell's equations in an extremely elegant, concise form.

Before we begin applying relativity to electromagnetism, we must define some mathematical machinery. First, we define a four-vector to be a set of four elements that transform as

$$\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu a^\nu, \tag{1}$$

where the Λ^μ_ν (where μ labels the row and ν labels the column) are the components of the Lorentz transformation matrix:

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

We also define two different kinds of four-vectors: covariant vectors (a_μ) and contravariant vectors (a^μ). Covariant and contravariant vectors differ from each other only in the sign of the 0^{th} component. We define the four-vector scalar product as

$$a_\mu b^\mu, \tag{3}$$

where we have invoked the Einstein summation convention, which states a summation is implied whenever a Greek index is repeated in a product, once as a covariant index and once as a contravariant index.

The last mathematical object we will need is a second-rank antisymmetrical tensor. In analogy with the relationship between the way a second rank tensor and a normal vector transform under coordinate rotations (two direction cosine factors as opposed to one), we can state the transformation rule for a second rank tensor under Lorentz transformations. For an arbitrary tensor $t^{\mu\nu}$, we have

$$\bar{t}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda\sigma} \tag{4}$$

In matrix form, the components of a four dimensional second rank tensor are

$$t^{\mu\nu} = \left\{ \begin{array}{cccc} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{array} \right\}. \tag{5}$$

An antisymmetric tensor obeys the relation

$$t^{\mu\nu} = -t^{\nu\mu} \tag{6}$$

For antisymmetric tensors, equation (5) reduces to

$$t^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{array} \right\}. \tag{7}$$

Note that there are only six independent components in this case. Applying equation (4) to the six components leads us to the following transformation rules:

$$\begin{array}{lll} \bar{t}^{01} = t^{01}, & \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}), & \bar{t}^{03} = \gamma(t^{03} + \beta t^{31}) \\ \bar{t}^{23} = t^{23}, & \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}), & \bar{t}^{12} = \gamma(t^{12} - \beta t^{02}). \end{array} \tag{8}$$

So now that we know about four-vectors and tensors, what can we say about the fields **E** and **B**? It can be shown by considering Lorentz contraction and time dilation that the fields transform via the following relations:

$$\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{array} \tag{9}$$

Comparing these transformation rules with equation (8), we can see that the fields must combine to form a second rank antisymmetrical tensor. Equating the first lines and the second lines of these two sets of transformations, we arrive at the field tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{array} \right\}. \tag{10}$$

Alternatively, we can equate the first line of equation (8) with the second line of equation (9) and vice versa to obtain the dual tensor $G^{\mu\nu}$:

$$G^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{array} \right\}. \tag{11}$$

Both of these tensors give the proper transformation rules. At this point, we can write down the relativistic Lorentz force law in terms of the field tensor and the proper velocity η^μ:

$$K^\mu = q\eta_\nu F^{\mu\nu}, \tag{12}$$

given that electric charge is a relativistic invariant.

We can also create field tensors for electromagnetism in matter by using the trick we found in chapter 6 whereby we can replace **B** with **D** and **E** with **H**. Our transformations become

$$\begin{array}{lll} \bar{H}_x = H_x, & \bar{H}_y = \gamma(H_y - vD_z), & \bar{H}_z = \gamma(H_z + vD_y), \\ \bar{D}_x = D_x, & \bar{D}_y = \gamma\left(D_y + \frac{v}{c^2}H_z\right), & \bar{D}_z = \gamma\left(D_z - \frac{v}{c^2}H_y\right). \end{array} \tag{13}$$

Applying the same logic as above, we arrive at the tensor $D^{\mu\nu}$:

$$D^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & -H_z & H_y \\ -cD_y & H_z & 0 & -H_x \\ -cD_z & -H_y & H_x & 0 \end{array} \right\}. \tag{14}$$

We can also construct its dual tensor $H^{\mu\nu}$:

$$H^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & H_x & H_y & H_z \\ -H_x & 0 & cD_z & -cD_y \\ -H_y & -cD_z & 0 & cD_x \\ -H_z & cD_y & -cD_x & 0 \end{array} \right\}. \tag{15}$$

Just as we have constitutive relations for the nonrelativistic fields in linear media, Minkowski proposed the followign constitutive relations for the field tensors:

$$D^{\mu\nu}\eta_\nu = c^2\epsilon F^{\mu\nu}\eta_\nu, \qquad H^{\mu\nu}\eta_\nu = \frac{1}{\mu}G^{\mu\nu}\eta_\nu, \tag{16}$$

where ϵ and μ are the proper permittivity and permeability, respectively, and η^μ is the four-velocity of the material. If the material is at rest, γ is 1 and β is 0, and so the field tensor $D^{\mu\nu}$ reduces to $\mathbf{D}$, $F^{\mu\nu}$ reduces to $\mathbf{E}$, $h^{\mu\nu}$ reduces to $\mathbf{H}$, and $G^{\mu\nu}$ reduces to $\mathbf{B}$. We therefore recover the proper non-relativistic forms of these constitutive relations.

In order to reformulate Maxwell's laws in this new relativistic tensor notation, we must first determine how the sources ρ and $\mathbf{J}$ transform. ρ depends on the volume of a given charge distribution. For any given direction of motion, one dimension of the volume will experience a Lorentz contraction. The current density in turn depends on ρ, and so it can be shown that the charge and current densities together form a current density four vector defined by

$$J^\mu = (c\rho, J_x, J_y, J_z). \tag{17}$$

In terms of this new quantity, the continuity equation reduces to

$$\frac{\partial J^\mu}{\partial x^\mu} = 0, \tag{18}$$

where a summation over μ is implied. This four vector along with the field tensor and the dual tensor allows us to rewrite Maxwell's equations in a more concise form:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0, \tag{19}$$

summing over ν. We can also write down Maxwell's equations in matter, using the $D^{\mu\nu}$ tensor we found above and our constitutive relations:

$$\frac{\partial D^{\mu\nu}}{\partial x^\nu} = J_f^\mu \tag{20}$$

Finally, we would like to be able to formulate relativistic versions of the potentials, since there are so many advantages to using the potential formulation. As one might expect, the scalar and vector potentials together form a four vector:

$$A^\mu = (\frac{V}{c}, A_x, A_y, A_z) \tag{21}$$

As we would hope, we can write the field tensor in terms of the four vector potential:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}. \tag{22}$$

This formulation immediately works with the homogeneous Maxwell equation, but the inhomogeneous equation becomes

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu. \tag{23}$$

We can render this equation into a more elegant form by noting that we still have the same gauge invariance with the four vector potential as we did with the normal potentials. The Lorentz gauge is (not surprisingly) a good gauge to choose, and in relativistic notation it becomes

$$\frac{\partial A^\mu}{\partial x^\nu} = 0. \tag{24}$$

Using this gauge, then, we can rewrite equation (23) as

$$\Box^2 A^\mu = -\mu_0 J^\mu. \tag{25}$$

This is the most elegant formulation of Maxwell's equations, and, as it contains all the information they do, encompasses all of classical electrodynamics when paired with the Lorentz force law, equation (12).