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Physics 113

The Electromagnetic Lagrangian and Hamiltonian

Up to this point, we have only applied the Lagrangian and Hamiltonian formalisms to velocity independent forces and potentials. When we introduce a velocity dependence, we can no longer say that

$$L = T - V,$$

where L is the Lagrangian, T is the kinetic energy, and V is the potential energy. We must redefine the Lagrangian to be the function that gives the right answer when fed into the Euler-Lagrange equation.

As an example of this, let us consider a charged particle moving in an electromagnetic field. The particle feels a force given by the Lorentz force law:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{1}$$

where we have used cgs units. Note that q is the electric charge of the particle and not a generalized coordinate.

Maxwell's equations and vector calculus let us express the electric and magnetic fields in terms of the scalar potential ϕ and the vector potential $\mathbf{A}$:

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned}$$

Inserting these expressions into the Lorentz force law, we get

$$\mathbf{F} = q\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v} \times (\nabla \times \mathbf{A})\right) \tag{2}$$

We now use the vector identity

$$\mathbf{B} \times (\nabla \times \mathbf{C}) = \nabla(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \nabla)\mathbf{C} - (\mathbf{C} \cdot \nabla)\mathbf{B} - \mathbf{C} \times (\nabla \times \mathbf{B}) \tag{3}$$

and the fact that $\mathbf{v}$ is not an explicit function of position to write

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}. \tag{4}$$

Now, writing the total derivative of the vector potential as

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{A}}{\partial z}\frac{dz}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}, \tag{5}$$

we can put the Lorentz force law into the form

$$\mathbf{F} = q\left(-\nabla\phi + \frac{1}{c}\nabla(\mathbf{v} \cdot \mathbf{A}) - \frac{1}{c}\frac{d\mathbf{A}}{dt}\right). \tag{6}$$

Now, even though we cannot say that $L = T - V$ because of the velocity dependence, there is a more general form of the Euler-Lagrange equation we can use. The kinetic energy is defined by

$$T = \sum_i \frac{1}{2} m_i \dot{x}_i^2.$$

Putting this into Newton's second law, we obtain

$$F_i = \frac{d}{dt}(m_i \dot{x}_i) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right). \tag{7}$$

If we now replace the x_i with generalized coordinates q_i, some algebra shows that

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_j m_j \dot{x}_j \left(\frac{\partial x_j}{\partial q_i}\right). \tag{8}$$

Taking a time derivative, more algebra shows that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = Q_i + \frac{\partial T}{\partial q_i}, \tag{9}$$

where the Q_i are the components of the generalized force and are given by

$$Q_i = \sum_j F_j \frac{\partial x_j}{\partial q_i}. \tag{10}$$

Rearranging equation (9), we get

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}, \tag{11}$$

which is the most general form of the Euler-Lagrange equation. In Cartesian coordinates, it reduces to

$$F_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i}.$$

If the system is conservative, the we can write

$$\mathbf{F} = -\nabla V,$$

and we come up with

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = 0, \tag{12}$$

which is the familiar Euler-Lagrange equation for $L = T - V$. If, however, the potential (call it U) is velocity-dependent, we can still write an equation of the exact same form as equation (12) with $L = T - U$ if the velocity-dependent force is of the form

$$F_j = -\frac{\partial U}{\partial x_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_j}\right). \tag{13}$$

We can put the Lorentz force law into this form by being clever. First, we write

$$\frac{dA_j}{dt} = \frac{d}{dt}\left(\frac{\partial}{\partial v_j}(\mathbf{v} \cdot \mathbf{A})\right),$$

since the partial derivative will pick out only the j^{th} component of the dot product. Now, since the scalar potential is independent of the velocity, we can add on a term containing it inside the partial derivative:

$$\frac{dA_j}{dt} = \frac{d}{dt}\left(\frac{\partial}{\partial v_j}(\mathbf{v} \cdot \mathbf{A} - q\phi)\right).$$

This lets us write the Lorentz force law as

$$F_x = -\frac{\partial}{\partial x}\left(q\phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A})\right) + \frac{d}{dt}\left(\frac{\partial}{\partial v_x}\left(q\phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A})\right)\right) \tag{14}$$

which in turn tells us that our generalized potential must be

$$U = q\phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A}). \tag{15}$$

This finally gives us the electromagnetic Lagrangian:

$$L = T - U = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} - q\phi + \frac{q}{c}(\mathbf{v}\cdot\mathbf{A}). \tag{16}$$

The Lagrangian in turn gives us the canonical momentum:

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + \frac{q}{c}\mathbf{A}. \tag{17}$$

Using the canonical momentum and the Lagrangian, we can construct the electromagnetic Hamiltonian:

$$H = \mathbf{p}\cdot\mathbf{v} - L = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi. \tag{18}$$