Legendre Polynomials in E&M

or

“Mr. Legendre Had Too Much Free Time”

The Polynomials

We begin with Legendre’s differential equation:

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \ell (\ell + 1) y = 0$$

We can solve this using the Method of Frobenius. First, we let

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

The differential equation then becomes

$$(1 - x^2) \left[ \sum_{n=0}^{\infty} (n+s) (n+s-1) a_n x^{n+s-2} \right] - 2x \left[ \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \right] + \ell (\ell + 1) \left[ \sum_{n=0}^{\infty} a_n x^{n+s} \right] = 0$$

Grouping terms together, we get

$$\sum_{n=0}^{\infty} \left[ (n+s) (n+s-1) a_n x^{n+s-2} - (n+s) (n+s-1) a_n x^{n+s} - 2(n+s) a_n x^{n+s} + \ell (\ell + 1) a_n x^{n+s} \right] = 0$$

Now, writing out the series, we have

$$[a_0 s (s-1)] x^{s-2} + [a_1 s (s+1)] x^{s-1} + [a_2 (s+2) (s+1) - a_0 s (s-1) - 2a_0 s + a_0 \ell (\ell + 1)] x^s + \cdots$$

$$+ [a_{n+2} (n+s+2) (n+s+1) - a_n (n+s) (n+s-1) - 2a_0 (n+s) + a_0 \ell (\ell + 1)] x^{n+s} + \cdots = 0$$

Looking at the first and second terms of this series and using the facts that $a_0 \neq 0$ by definition and that each of the coefficients of the series must be independently 0, we have

$$s^2 - s = 0 \Rightarrow s = 0, 1$$

$$a_1 s (s+1) = 0 \Rightarrow s = 1, a_1 = 0$$

$$s = 0, a_1 \neq 0$$

Looking at the $n^{th}$ term of the series, we obtain the recursion relation:

$$a_{n+2} = \frac{(n+s) (n+s-1) + 2(n+s) - \ell (\ell + 1)}{(n+s+2) (n+s+1)} a_n$$

Using our two values for $s$, we arrive at two different relations:

$$s = 0 : a_{n+2} = \frac{n (n-1) + 2n - \ell (\ell + 1)}{(n+2) (n+1)} a_n$$

$$s = 1 : a_{n+2} = \frac{n (n+1) + 2(n+1) - \ell (\ell + 1)}{(n+3) (n+2)} a_n$$
This finally provides us with two linearly independent series solutions, one containing only even powers of \( x \), the other only odd powers:

\[
y_1(x) = 1 - \frac{\ell(\ell + 1)}{2!} x^2 + \frac{\ell(\ell + 1)^2 - 6\ell(\ell + 1)}{4!} x^4 + \cdots \\
y_2(x) = x + \frac{2 - \ell(\ell + 1)}{3!} x^3 + \frac{24 - 14\ell(\ell + 1) + (\ell(\ell + 1))^2}{5!} x^5 + \cdots
\]

For all values of \( \ell \), one of the above series will not terminate. The non-terminating series are called Legendre functions of the second kind.

When the terminating series (which becomes a polynomial in \( x \)) is multiplied by a particular constant (in order to make \( P_\ell(1) = 1 \)), it becomes one of the Legendre polynomials. The Legendre polynomials are conveniently given by Rodrigue’s Formula:

\[
P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell
\]

The Legendre polynomials may also be obtained by means of a generating function:

\[
\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{\ell = 0}^\infty P_\ell(x) t^\ell
\]

The Legendre polynomials are a complete set of functions, and as such obey an orthogonality relation:

\[
\int_{-1}^{1} P_\ell(x) P_\ell'(x) dx = \frac{2}{2\ell + 1} \delta_{\ell\ell'}
\]

Using this orthogonality relation, if we have a function \( f(x) \) defined by

\[
f(x) = \sum_{\ell = 0}^\infty A_\ell P_\ell(x)
\]

then

\[
A_\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} f(x) P_\ell(x) dx
\]

It is also interesting to note that the integral of a Legendre polynomial can be expressed in terms of other Legendre polynomials:

\[
\int P_\ell(x) = \frac{P_{\ell+1}(x) - P_{\ell-1}(x)}{2\ell + 1}
\]

**Physical Applications**

Laplace’s Equation in Spherical Coordinates:

When the separation of variables technique is applied to Laplace’s equation of a spherical, azimuthally symmetric charge distribution, one finds that the ODE for the angle \( \theta \) is

\[
\frac{d}{d\bar{\theta}} \left( \sin \theta \frac{d\Theta}{d\bar{\theta}} \right) = -\ell (\ell + 1) \sin \theta \Theta
\]


if we let \( x = \cos \theta \) and rewrite this equation, we arrive at

\[
(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d \Theta}{dx} + \ell (\ell + 1) \Theta = 0
\]

which is exactly Legendre’s equation. Thus, solutions to the angular part of Laplace’s equation are Legendre polynomials in the variable \( \cos \theta \):

\[
\Theta (\theta) = P_\ell (\cos \theta)
\]

Multipole Expansions:

The potential of an arbitrary charge distribution at a point outside the distribution is given by

\[
V (r) = \frac{1}{4 \pi \epsilon_0} \int \frac{1}{r} \rho (r') \, d\tau'
\]

Using the law of cosines, we can express as

\[
= r \sqrt{1 + \left( \frac{r'}{r} \right)^2 - 2 \left( \frac{r'}{r} \right) \cos \theta'}
\]

and so we see that \( \frac{1}{r} \) is the generating function for the Legendre polynomials, using \( x = \cos \theta' \), and so

\[
\frac{1}{r} = \sum_{\ell=0}^{\infty} \frac{r'}{r^{\ell+1}} P_\ell (\cos \theta')
\]