The Inertia Tensor

In studying the mechanics of rigid bodies, a funny object called the inertia tensor appears. Though for all intents and purposes it appears to be simply a three by three matrix, it is actually a second rank tensor, as can be shown by examining how it behaves under coordinate transformations. Since the inertia tensor is a Cartesian tensor, we will consider only rotations in Cartesian space.

Let us begin with the relation connecting angular momentum and angular velocity:

\[ L_k = \sum_i I_{ki} \omega_i, \]  

where \( I_{ki} \) are the components of the inertia tensor. The fact that these components are indeed the components of a second rank tensor can be shown by applying the general quotient rule for tensors, which states that if the relation

\[ A_{pq...k...m} B_{ij...k...m} = C_{pq...mij...n} \]  

holds for all rotated coordinate systems and if \( \mathbf{B} \) is an \( m^{\text{th}} \) rank tensor and \( \mathbf{C} \) is an \( n^{\text{th}} \) rank tensor then \( \mathbf{A} \) is an \( (m - n + 2)^{\text{th}} \) rank tensor. Since the angular momentum and angular velocity are rank one tensors, the \( I_{ki} \) must be the components of a second rank tensor.

Since equation (1) is a vector equation, it must have a completely analogous form under a coordinate rotation:

\[ L'_i = \sum_j I'_{ij} \omega'_j \]  

where the primed quantities are measured with respect to the rotated coordinate system. We now apply the transformation rules for vectors derived in chapter one:

\[ L_k = \sum_{m} \lambda_{mk} L'_m \]  
\[ \omega_l = \sum_{j} \lambda_{jl} \omega'_j \]  

Substituting back into equation (3), multiplying by \( \lambda_{ik} \), and summing over \( k \), we have

\[ \sum_{m} \left( \sum_k \lambda_{ik} \lambda_{mk} \right) L'_m = \sum_{j} \left( \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \right) \omega'_j \]  

Given the properties of direction cosines, the summation over \( k \) on the left side is simply \( \delta_{im} \), and so we have

\[ L'_i = \sum_{j} \left( \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \right) \omega'_j \]  

Equating this with equation (3), we find that

\[ I'_{ij} = \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \]  

which is the general transformation rule for second rank tensors.
We now use the fact that a second rank Cartesian tensor is mathematically equivalent to a matrix (note that this is not true for tensors in curvilinear space or for higher rank tensors!) to manipulate the inertia tensor into a nicer form. First, we rewrite equation (8) as

\[ I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda_{lj}^{T} \]  

(9)

We now rewrite equation (9) as a matrix equation:

\[ I' = \lambda I \lambda^{T} \]  

(10)

Noting that since the \( \lambda \) matrices are orthogonal so that \( \lambda^{T} = \lambda^{-1} \), we arrive at the similarity transformation

\[ I' = \lambda I \lambda^{-1} \]  

(11)

We can use this similarity transformation to diagonalize the inertia tensor and find its eigenvalues and eigenvectors. We need to find a rotation such that the new tensor is diagonal, which we can express with the Kronecker delta:

\[ I_{i} \delta_{ij} = \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \]  

(12)

Multiplying this equation by \( \lambda_{im} \) and summing over \( i \) we obtain

\[ \sum_{i} I_{im} \delta_{ij} \lambda_{im} = \sum_{k,l} \left( \sum_{i} \lambda_{im} \lambda_{ik} \right) \lambda_{jl} I_{kl} \]  

(13)

We note that the term in parentheses on the right side of this equation is just \( \delta_{mk} \). Some algebra leads to the following equation:

\[ \sum_{l} (I_{ml} - I_{j} \delta_{ml}) \lambda_{jl} = 0 \]  

(14)

which is simply a set of simultaneous linear equations. For a nontrivial solution to these equations to exist, the determinant of the coefficients must vanish, and so we have

\[ |I_{ml} - I_{\delta_{ml}}| = 0. \]  

(15)

Equation (15) is the secular determinant of \( I \), and is a cubic equation. The three roots \( I_{1}, I_{2}, \) and \( I_{3} \) are the eigenvalues of the diagonalized tensor, and are called the principal moments of inertia. The new diagonalized coordinate axes of the diagonalized tensor are its eigenvectors, and are called the principal axes. Due to the fact that the inertia tensor is a Hermitean tensor (i.e., \( I_{ij} = I_{ji}^{*} \)), both its eigenvectors and eigenvalues will be real.