Nick Ouellette
Physics 111

Hohmann Transfers and Gravity Assists

In 1925, Walter Hohmann found the most energy efficient method of transferring between two planetary orbits. Hohmann transfers require two energy expenditures (rocket thrusts), one at each end. The method is described here for a transfer between Earth and Mars. First of all, we can approximate the orbits of Earth and Mars by circles, since their eccentricities are very small. We want to find, then, the ellipse connecting these two circular orbits, where Earth is at the perihelion and Mars is at the aphelion. We need to fire the rockets to accelerate to some v_{t1} to leave Earth's circular orbit and once more to decelerate to v_{t2} to settle into Mars's circular orbit. We will consider only the gravitational force from the Sun and disregard that of Earth and Mars. Let r_1 be the perihelion of the transfer orbit and r_2 be the aphelion. Also, let v_1 be the velocity in Earth's circular orbit and v_2 be the velocity in Mars's orbit.

Using equation 8.42, we can find the energy of the initial orbit, we have

$$E = \frac{-k}{2r_1} = \frac{1}{2}mv_1^2 - \frac{k}{r_1} \tag{1}$$

using the fact that $E = T + U$. This gives

$$v_1 = \sqrt{\frac{k}{mr_1}} \tag{2}$$

Using this same method, we can calculate the perihelion energy of the transfer orbit, E_t:

$$E_t = \frac{-k}{r_1 + r_2} = \frac{1}{2}mv_{t1}^2 - \frac{k}{r_1} \tag{3}$$

Solving for v_{t1}, we have

$$v_{t1} = \sqrt{\frac{2k}{mr_1}\left(\frac{r_2}{r_1 + r_2}\right)} \tag{4}$$

The change in velocity we need from the rockets is just the difference of these two velocities:

$$\Delta v_1 = v_{t1} - v_1 \tag{5}$$

Using a similar method, we can obtain the velocities for the aphelion of the transfer orbit:

$$v_2 = \sqrt{\frac{k}{mr_2}} \tag{6}$$

$$v_{t2} = \sqrt{\frac{2}{m}\left(E_t + \frac{k}{r_2}\right)} = \sqrt{\frac{2k}{mr_2}\left(\frac{r_1}{r_1 + r_2}\right)} \tag{7}$$

and

$$\Delta v_2 = v_2 - v_{t2} \tag{8}$$

The total time it takes to complete the Hohmann transfer will be one half the period of the transfer orbit. Using equation 8.48, we have

$$T_t = \frac{\tau_t}{2} = \pi\sqrt{\frac{m}{k}}\left(\frac{r_1 + r_2}{2}\right)^{\frac{3}{2}} \tag{9}$$

To perform a Hohmann transfer from Earth to any of the outer planets, the spacecraft should be launched in the direction of the Earth's orbit; to be transferred to an inner planet, the spacecraft should be launched opposite the Earth's orbit. Note that while a Hohmann transfer uses a minimum of energy, it is the not the shortest path in time. To decrease the time of a transfer orbit, we can either use more fuel or employ a gravity assist.

A gravity boost occurs when a spacecraft follows a transfer orbit to a point on an object's orbit *behind* the object. When the spacecraft reaches the object's orbit, it will be pulled by the object's gravitational field and will fall toward the object, increasing its velocity (and decreasing the velocity of the object, but we assume that the mass difference is great enough that the object's change in velocity is negligible).

As an example, consider a spacecraft performing a gravity assist around Jupiter. Relative to Jupiter, the spacecraft approaches it with some velocity (assumed to be greater than Jupiter's escape velocity, giving it a nonnegative total energy, and thus an open orbit). As the spacecraft falls toward Jupiter, its speed will increase. Once the spacecraft starts moving back away from Jupiter, its speed will begin to decrease again, and from the perspective of Jupiter the direction and not the magnitude of the velocity will have changed. But relative to the Sun, the spacecraft will have been dragged along with Jupiter and will have gained some angular momentum, and will have therefore also added to its kinetic (and therefore total) energy.

We can use a variant of this process to slow spacecraft as well (gravity brakes). If instead of creating a transfer orbit such that the spacecraft enters the planet's orbit behind the planet we plan for the spacecraft to arrive in front of the planet, the velocity of the spacecraft will be decreased. In this way, we could send a space craft to Mercury by using a gravity brake around Venus.