Fermions and Bosons

Consider a wavefunction $\psi(r_1, r_2)$ describing a system of two identical particles. Quantum mechanics tells us that only the absolute square of the wavefunction is physically measurable, and so using the fact that the particles are indistinguishable, we can write

$$|\psi(r_1, r_2)|^2 = |\psi(r_2, r_1)|^2.$$ 

This implies two different classes of solutions for the wavefunction; some solutions have

$$\psi(r_1, r_2) = \psi(r_2, r_1)$$

and are symmetric, but antisymmetric solutions where

$$\psi(r_1, r_2) = -\psi(r_2, r_1)$$

are also allowed. Now, let $\alpha$ and $\beta$ be sets of quantum numbers. We can write general symmetric and antisymmetric solutions as follows:

$$\psi_S(r_1, r_2) = \frac{1}{\sqrt{2}} (\psi_\alpha(r_1)\psi_\beta(r_2) + \psi_\alpha(r_2)\psi_\beta(r_1))$$

is the symmetric solution and

$$\psi_A(r_1, r_2) = \frac{1}{\sqrt{2}} (\psi_\alpha(r_1)\psi_\beta(r_2) - \psi_\alpha(r_2)\psi_\beta(r_1))$$

is the antisymmetric solution. Particles that have antisymmetric wavefunctions are fermions, and particles with symmetric wavefunctions are bosons. We can immediately see that the Pauli exclusion principle will apply to fermions by examining the antisymmetric wavefunction. If $\alpha = \beta$, which would put the two particles in the same quantum state, $\psi = 0$, telling us that there is no probability of finding a fermionic system where any particles are in the same state. Bosons clearly do not obey the exclusion principle, and in fact are quite happy being in the same state. This difference can also be seen in comparing the average state occupation numbers for fermions and bosons. Fermions have

$$\langle n_a \rangle_{FD} = \frac{1}{e^{\beta(\epsilon_a - \mu)} + 1},$$

implying that $\langle n_a \rangle_{FD} \leq 1$, and bosons have

$$\langle n_a \rangle_{BE} = \frac{1}{e^{\beta(\epsilon_a - \mu)} - 1},$$

implying that $\langle n_a \rangle_{BE} \leq N$, meaning that there can only be one fermion in a given state but as many bosons as we’ve got in a single state.

On a single particle level, fermions and bosons are differentiated by their spin. Fermion spin comes in half integer multiples of $\hbar$, while boson spin comes in full integer multiples. Using the fact that protons, neutrons, and electrons are all fermions, we can see that the He$^3$ atom must be a fermion (adding up an odd number of half integers gives a half integer) but the He$^4$ atom is a boson (summing an even number of half integers gives a whole integer).