Plasma Waves

Dispersion Relations
Waves of different frequency travel at different speeds in a dispersive medium, and the frequency ω can be expressed as a function of the wave number k. This is known as the dispersion relation. For example, light waves in a vacuum have the dispersion relation $\omega = ck$, and sound waves also have a linear dispersion relation.

In plasmas, there are a variety of waves which propagate with a variety of nonlinear dispersion relations. We will examine a few of these waves and the approximations needed to find their dispersion relations. For more information, see Goldston and Rutherford’s Plasma Physics.

Plasma Oscillation: A Simple Model
Consider a section of plasma of cross-section $A$ in which the electrons have been displaced from their equilibrium condition by a distance $\delta$, leaving the ions unmoved. The electric field in the plasma is like the field inside a parallel plate capacitor, $E = \sigma/\varepsilon_0 = en_e \delta$, where $e$ is the electron charge and $n_e$ is the number density of electrons. The force on the electrons due to this field is given by $F = Q_{\text{total}}E = (en_e \delta A)$.

By Newton’s Second Law, the force can also be written as $F = M_{\text{total}}\ddot{\delta} = m_e n_e \delta A \ddot{\delta}$.

Combining Eqs. (2) and (3) gives simple harmonic motion: $\ddot{\delta} = \omega_p^2 \delta$, $\omega_p = \left(\frac{n_e e^2}{m_e \varepsilon_0}\right)^{\frac{1}{2}}$

The electric field pulls the electrons back towards equilibrium, where they exactly neutralize the ion charge, but the kinetic energy gained in this process causes the electrons to overshoot to a new displacement on the other side.

Langmuir Waves
We will now perform a more sophisticated analysis of the same situation. Adding a term for the scalar electron pressure $(\nabla p_e)$ to the net force and performing the chain rule to find $\dot{n}_e$, where $\mathbf{u}_e$ is the velocity, Newton’s Second Law becomes

$m_e n_e [\dot{\mathbf{u}}_e + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e] = -en_e \mathbf{E} - \nabla p_e$

We make the assumption that $\mathbf{u}_e, n_e, \mathbf{E}$, and $p_e$ each have a constant part (like $\mathbf{u}_0$) added to a very small oscillating part (like $\mathbf{u}_1$). If the oscillations are assumed to be harmonic and in the $x$ direction (so that the oscillating parts have the form $e^{i(kx - \omega t)}$, the $\nabla$ operator becomes $ik$, and $\frac{\partial}{\partial t}$ becomes $-i\omega$. Letting $\omega_0 = 0$, $E_0 = 0$, and neglecting second-order terms, Eq. (5) becomes $-i\omega m_e n_0 u_1 = -en_0 E_1 - ikp_1$

We can write the continuity equation (statement of local electron charge conservation) as

$\dot{n}_e + \nabla \cdot (n_e \mathbf{u}_e) = 0 \Rightarrow -i\omega n_1 + ik n_0 u_1 = 0$

To write Gauss’s Law, we also have to consider the ions, since they contribute to the E-field. Assuming they each have charge $+e$ and that the ion density is $n_i$, Gauss’s Law becomes

$\varepsilon_0 \nabla \cdot \mathbf{E} = e(n_i - n_e) \Rightarrow ik\varepsilon_0 E_1 = -en_1$
Substituting from Eqs. (7) and (8) into Eq. (6) gives

\[ \frac{i\omega^2 m_e n_1}{k} = -\frac{e^2 n_0 n_1}{ik\epsilon_0} + ikp_1 \]  

(9)

With the assumption that the electron compression occurs one-dimensionally and faster than thermal conduction, we have \( p_1 = \gamma T n_1 \), with \( \gamma = 3 \). Substituting this into Eq. (9) and rearranging, we have the Bohm-Gross dispersion relation for electrostatic plasma waves (Langmuir waves),

\[ \omega^2 = \frac{e^2 n_0}{m_e \epsilon_0} + \frac{3k^2 T}{m_e} = \omega_p^2 + 3kv_{t,e}^2 \]  

(10)

where \( v_{t,e} \) is the electron thermal velocity. We can trace this additional frequency component to the inclusion of electron pressure in our forces. Note that because of it, \( \omega \leq \omega_p \). Also, for long wavelengths (low \( k \)) or low temperature, the wave phase velocity \( \frac{\omega}{k} \) can become larger than \( v_{t,e} \), or even larger than \( c \) (which is fine, since energy travels at the group velocity, not the wave velocity). See attached Figure (1) for a graph of this dispersion relation.

Note that we can confirm that this situation is electrostatic by finding the first-order expression for \( J \),

\[ J = -en_0 u_1 = -\frac{e}{k}n_1 = i\omega_0 E_1 = -\epsilon_0 \dot{E}_1 \]  

(11)

and seeing that Ampère's law with Maxwell's correction goes to zero:

\[ \nabla \times B_1 = \mu_0 J + \mu_0 \epsilon_0 \dot{E}_1 = 0 \]  

(12)

**Ion Sound Waves**

Another electrostatic plasma wave arises from the motion of the ions, caused by a pressure of the form \( nkT \). The dispersion relation for these waves is shown in Figure (2). For small \( k \), it has the linear form of a normal sound wave, with a slope of the ion sound speed \( C_s = (T_e/M)^{1/2} \). Compare the graph of this dispersion relation with the Langmuir wave dispersion relation.

**The Dielectric Tensor**

More plasma waves arise when we consider a background magnetic field, \( B_0 \). When the waves propagate perpendicular to the magnetic field (\( k \perp B_0 \)), we see ordinary waves (O-waves) whose electric fields are oriented along the magnetic field (\( E_1 \parallel B_0 \)) and extraordinary waves (X-waves) with \( E_1 \perp B_0 \). When the waves propagate along the magnetic field (\( k \parallel B_0 \)) and when \( E_1 \parallel B_0 \), we see the Langmuir waves described earlier, but when \( E_1 \perp B_0 \), we have left and right circularly polarized waves (L and R waves).

Using tensor notation, all of these waves can be considered together. The fluid equation of motion (making the same approximation as before that each quantity has a constant part and a very small oscillating part) can be written as

\[ mn_0 \frac{\partial u_1}{\partial t} = qn_0 (E_1 + u_1 \times B_0) - \gamma T \nabla n_1 \]  

(13)

(Compare to Eq. (5).) Making the same approximation of harmonic oscillations in each dimension, and again linearizing the continuity equation, we obtain a set of linear equations for the components of \( u_1 \). We then use Ohm's law to find the electrical conductivity tensor:

\[ J_1 = \sum n_0 q u_1 = \sigma \cdot E_1 \]  

(14)

The tensor conductivity can be substituted into the wave equation, and we derive

\[ (\omega^2 \mu_0 \varepsilon - k^2 \chi) \cdot E_1 = 0 \]  

(15)

where \( \varepsilon \) is the dielectric tensor, given by

\[ \varepsilon = \epsilon_0 \left( 1 + \frac{i\sigma}{\epsilon_0 \omega} \right) \]  

(16)
and $X$ is the tensor defined by $X = I - \mathbf{k}\mathbf{k}/k^2$, which, letting $\theta$ be the angle between $\mathbf{k}$ and $\mathbf{B}_0$ and choosing $k_y = 0$ so that $\mathbf{k} = k \sin \theta \mathbf{x} + k \cos \theta \mathbf{z}$, looks like

$$
X = \begin{pmatrix}
\cos^2 \theta & 0 & -\sin \theta \cos \theta \\
0 & 1 & 0 \\
-\sin \theta \cos \theta & 0 & \sin^2 \theta
\end{pmatrix}
$$

(17)

The dispersion relation can be found by setting the determinant of the tensor quantity in parenthesis in Eq. (15) equal to zero.

**Magnetohydrodynamic Oscillation**

According to Alfvén’s Theorem (remember my presentation two weeks ago?), the magnetic flux through any closed loop moving with a plasma is constant in time, which means that the plasma is dragged around with the field lines. If some perturbation, such as a current loop, causes some field lines to be pulled together, this generates a pressure of the form $B^2/2\mu_0$, and the movement of the plasma to relieve this pressure results in another oscillation - Alfvén waves. We can find the characteristic speed of these waves by setting the energy density $\frac{1}{2}\rho v^2$ equal to the pressure density $\frac{B^2}{2\mu_0}$ to get

$$
v_A = \left( \frac{B^2}{\rho\mu_0} \right)^{\frac{1}{2}}
$$

(18)