Development and Applications of Wavelets in Signal Processing

Mathematics 097: Senior Conference Paper
Published May 2014

David Nahmias
dnahmias1@gmail.com

Abstract

Wavelets have many powerful applications in signal processing. We look to explain the derivation and use of two Daubechies wavelets, the Daubechies-2 wavelet, or otherwise known as the Haar wavelet, and the Daubechies-4 wavelet. We begin by examining other existing methods of signal processing. In particular we will look first look at the discrete Fourier transform as well as the windowed Fourier transform as methods of signal analysis. However, as we will see, these methods do not characterize signals that change with both frequency and time well. We will look then to the method of multiresolutional analysis as a means to solve many of the problems found in these other methods. From the definition of a multiresolutional analysis we find a general form for wavelet transforms thus beginning our construction of the discrete wavelet transform. We then go on to define the general form of discrete orthogonal wavelets and derive the wavelet coefficients for the Haar wavelet and the Daubechies-4 wavelet. Finally, we show how the wavelet transform is used in signal processing by stepping through an application of the wavelet transform.

Department of Mathematics, Swarthmore College

1. Introduction

French mathematician Yves Meyer said, "Often pure math 'trickles down' to applications, but this was not the case for wavelets. This is not something imposed by the mathematicians; it came from engineering. I recognized familiar mathematics, but the scientific movement was from application to theory."[1] This is perhaps one of the most intriguing things about the development of wavelets. However, before continuing with our discussion of wavelets, let us delve into some of the history of the field.

In 1807, another French mathematician Joseph Fourier showed that any function can be expressed as a the sum of sines and/or cosines. This at the time came as a mathematical revolution. It turned many problems that were at the time unsolvable and simplified them to solvable ones. However, it is clear that theoretically to reconstruct a function, especially irregular, perfectly would require the computation of an infinite number of coefficients and infinite sum of these sinusoidal functions. Thus, an approximation or 'cut-off' to the number of terms used is needed if a numerical solution is to be obtained. This of course is a constraint to all numerical approximation methods. When considering only the discrete Fourier transform, as most of those in signal analysis do, we see the closed form formula

$$X[k] = \frac{1}{\sqrt{D}} \sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi}{D}kn} \tag{1}$$

where $x[n]$ is some discrete signal of n elements and $\frac{1}{\sqrt{D}}$ is the normalizing factor, sometimes referred to as the 'DC component' because it is a constant, or direct current, imposed on all elements of the signal.

Since this is a linear transformation we can represent the discrete Fourier transform as matrix operations such that

$$\mathbf{W} \times \overrightarrow{x[n]} = \begin{pmatrix} W_{0,0} & W_{0,1} & \cdots & W_{0,N-2} & W_{0,N-1} \\ W_{1,0} & W_{1,1} & \cdots & W_{1,N-2} & W_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{N-2,0} & W_{N-2,1} & \cdots & W_{N-2,N-2} & W_{N-2,N-1} \\ W_{N-1,0} & W_{N-1,1} & \cdots & W_{N-1,N-2} & W_{N-1,N-1} \end{pmatrix}_{n\times n} \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-2] \\ x[N-1] \end{pmatrix}_{n\times 1} = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-2] \\ X[N-1] \end{pmatrix}_{n\times 1}$$

where, $W_{k,n} = \frac{1}{\sqrt{D}} e^{-i\frac{2\pi}{D}kn}$

However, Fourier analysis has other limits effecting its ability to process signals. It is poorly suited to handle short signals or signals that have sudden or unpredictable changes. A signal that resembles a square wave will not be represented well by the Fourier transform, regardless the number of terms used. This is because the Gibbs Phenomenon ensures that there will be at least a 9% overshoot from the Fourier transform at the edges of the square wave. Furthermore, the Fourier transform makes it clear how much each frequency a signal contains but obfuscates when the various frequencies were emitted or for how long. Fourier analysis analyzes a signal at any given time identically to any other. Time information is not destroyed by the Fourier transform, it is simply absorbed in the phases output by the transform. The fact that information about any point in the signal is spread out over the entire transform, contained in all the frequencies of the transform, can be a serious drawback in analyzing signals. This lack of time information makes the Fourier transforms vulnerable to errors. If a relatively long signal has an error only at the very end, then the error will corrupt the entire Fourier transform. These errors can be in the form of phase errors, which can lead the Fourier transform to yield a result that does not accurately represent the original signal.

In an attempt to solve these problems, in the 1940s the windowed Fourier transform, or sometimes called the short-time Fourier transform, was created. The idea of this modified Fourier transform is to analyze the frequencies of a signal by sections. This way, whatever the result of the Fourier transform, it could be traced back to some specific section of the signal where the analysis took place. The closed form equation for the windowed Fourier transform is

$$X[f_k, t_m] = \sum_{n=0}^{N-1} x[n]\omega[n-m]e^{i\frac{2\pi}{N}kn} \tag{2}$$

where f_k is the k^{th} discrete frequency and t_m is the starting time of the m^{th} window of analysis.

So while the traditional Fourier transform uses sines and cosines to analyze a signal, the windowed Fourier transform uses smaller pieces of the curve. This curve serves as the 'window,' which remains fixed in size for the analysis. In each 'window,' different oscillations of varying frequency are placed in order to represent the original signal. However, this fixed window size impose some limitations to the windowed Fourier transform. The smaller the window, the better the transform will be at representing high frequency changes, such as peaks or discontinuities, but it will be more blind to the lower-frequency components of the signal. This is simply because the lower frequencies might not fit into the windows of smaller size. However, if larger windows are used, then more of the low frequencies will be captured and present in the transform but high frequency changes may be lost and the ability to localize in time will diminish.

These flaws among others led to the desire and need for a signal analysis method that uses what is called a multiresolutional analysis method. This new method adapts automatically to the different components of a signal, using a big window to look at enduring components of low frequency and progressively smaller windows to look at short-lived components of high frequency. The signal is studied at a coarse resolution to get an overall picture and at higher and higher resolutions to see increasingly finer details.[1] Furthermore, discrete data often changes rapidly and at different rates unlike well behaved continuous functions. Such behaviors make Fourier analysis difficult to use in some circumstances of analyzing chaotic discrete signals while the wavelet transform turns out to be much better.

In the 1980s Ingrid Daubechies developed discrete orthogonal wavelets with compact support. Compactly supported wavelets have the value zero everywhere outside a certain desired interval. This series of wavelets have since become know as Daubechies wavelets. These wavelets use the multiresolutional analysis approach in their application, which we will see. Specifically, we will delve into the derivation and use of the Daubechies-2, or Haar, wavelet and the Daubechies-4 wavelet transforms.

2. Background for Signal Processing and Discrete Transforms

The signals being discussed are those that one would usually think of as coming from a computer. As such, they are discrete because of how information is stored and so when talking about signal processing we refer to the discrete version of a transform or method of analysis. This was the case with the Fourier transforms previously outlined and it will be the case for the wavelet transforms.

As such, even though a signal may be stored or represented discretely, digitally, its source may in fact have been analogue or continuous. Thus, we say that these continuous signals are being sampled and state for reference Shannon's sampling theorem. This theorem proves that if the range of frequencies of a signal measured in hertz (cycles per second) is B the signal can be represented with complete accuracy by measuring its amplitude $2B$ times per second. Furthermore, as we will see with the nature of the wavelet transform, in accordance with Shannon's sampling theorem, wavelets automatically 'sample' high frequencies more often than low frequencies, since as the frequency doubles the number of wavelets doubles.

Furthermore, only orthogonal transformations will be discussed. The orthogonality of a system implies several facts about it which we will see give rise to their derivations. In addition to the many mathematical properties that come about from orthogonality we note that because we are talking about a discrete system where information is being represented and stored we see that orthogonality implies that no information is repeated. Families of orthogonal wavelets also form such systems. The vectors formed by a wavelet and all its translations and dilations are all perpendicular to each other. The speed of computing orthogonal wavelet coefficients is a consequence of this geometry. This is also true regarding the speed of calculating Fourier coefficients, since sines and cosines also form an orthogonal basis.

Finally, the functions, signals and spaces being considered are all said to be in the Hilbert Space $L^2(\mathbb{R})$. The $L^2(\mathbb{R})$ is the space of functions with norm, $||f(t)|| = \sqrt{\int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t}$. The important implication this space carries with regards to signal processing is that any function, or signal, $f(t)$, that is in $L^2(\mathbb{R})$ is said to be square integrable, $\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t < \pm\infty$. In the field of signal processing a signal is said to have finite power if this is the case. In order to preform any of the signal analysis methods described here, Fourier or wavelet, a signal must have finite power since all these methods are preforming a convolution which might diverge, thus not yielding a solution or sending a computer analysis to run indefinitely, if this condition is not met.

3. Construction of Discrete Orthonormal Wavelet Transforms: Multiresolutional Analysis

Using the definition of an orthogonal multiresolutional analysis we will now construct the general formula for the orthogonal wavelet transform.

An orthogonal multiresolution analysis consists of a sequence of closed subspaces $V_j, j \in \mathbb{Z}$ of $L^2(\mathbb{R})$ satisfying,

(i) $\{0\} \subset V_j \subset V_{j+1} \subset L^2(\mathbb{R})$, for all $j \in \mathbb{Z}$.
(ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ (trivial intersection).
(iii) $\overline{\bigcup_{j \in \mathbb{Z}}} V_j = L^2(\mathbb{R})$ or equivalently $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in the $L^2(\mathbb{R})$ (density).
(iv) $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$ (scaling property).
(v) $f(t) \in V_0$ if and only if $f(t-k) \in V_0$ for any $k \in \mathbb{Z}$ (translation invariance).
(vi) There exists a scaling function $\varphi \in V_0$ such that $\{\varphi(t-k) : k \in \mathbb{Z}\}$ is an orthonormal basis of

V_0.

We call φ the scaling function, or otherwise referred to as the Father wavelet. [5]

Let us first recall the definition of an orthogonal complement given that V is a subspace of $L^2(\mathbb{R})$. V's orthogonal complement $V^\perp = \{\overrightarrow{W} \in L^2(\mathbb{R}) : \overrightarrow{W} \cdot \overrightarrow{V} = 0 \text{ for every } \overrightarrow{V} \in V\}$.

Furthermore, if $\{L^2(\mathbb{R})_k : k \in \mathbb{Z}\}$ is a sequence of mutually orthogonal closed subspaces of $L^2(\mathbb{R})$ we let $V = \bigoplus_{k=-\infty}^{\infty} L^2(\mathbb{R})_k$ denote the closed subspace consisting of all $f(t) = \sum_{k\in\mathbb{Z}} f(t)_k$ with $f(t)_k \in L^2(\mathbb{R})_k$ and $\sum_{k\in\mathbb{Z}} ||f(t)_k||^2 < \infty$. We call V the orthogonal direct sum of the spaces $L^2(\mathbb{R})_k$.

We can now use these facts and our notation and definition of multiresolutional analysis to construct the orthonormal wavelet.

We let W_0 be the orthogonal complement of V_0 in V_1, that is, $V_1 = V_0 \bigoplus W_0$. Then, if we dilate the elements of W_0 by 2^j, we obtain a closed subspace W_j of V_{j+1} such that $V_{j+1} = V_j \bigoplus W_j$ for all $j \in \mathbb{Z}$.

Since $V_j \to \{0\}$ as $j \to -\infty$ we see that $V_{j+1} = V_j \bigoplus W_j = \bigoplus_{i=-\infty}^{j} W_i$ for all $j \in \mathbb{Z}$.

And since $V_j \to L^2(\mathbb{R})$ as $j \to \infty$ we see that $L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$.

Thus, to find an orthonormal wavelet , we need to find a function $\psi \in W_0$ such that $\{\psi(t-k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_0. Then, through a change of variable calculation we can see that $\{2^{j/2}\psi(2^j t - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for W_j for all $j \in \mathbb{Z}$ from the scaling property and our definition of W_j. Hence, $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, which shows that ψ is an orthonormal wavelet on $\mathbb{R}$.[3]

Thus we see,

Definition*: A wavelet is a function $\Psi(t) \in L_2(\mathbb{R})$ such that the family of functions*

$$\Psi_{j,k} =: 2^{j/2}\Psi(2^j t - k), \tag{3}$$

where j and k are arbitrary integers, in an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. [7]

This function, or family of functions, Ψ is called the wavelet function, and is also referred to as the Mother wavelet. It is dilated and translated through the signal by the j, k indexes creating the desired multiresolutional analysis.

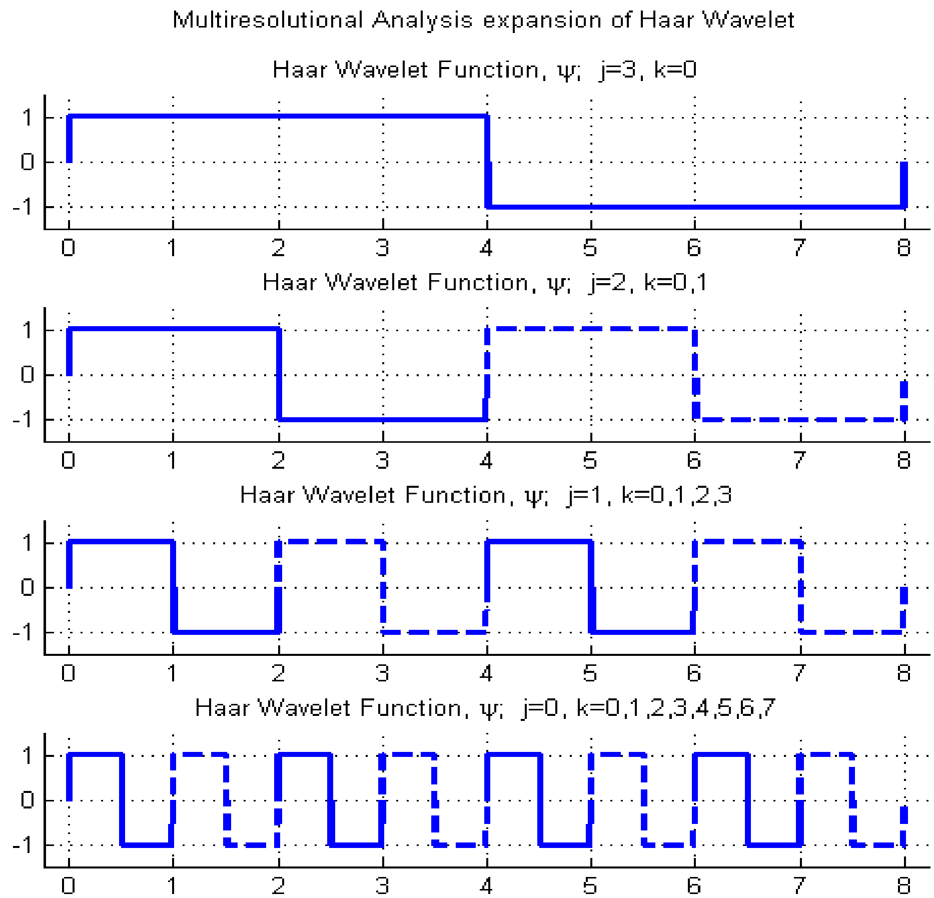

Figure 1: *Multiresolutional Analysis expansion of some possible j, k values and their corresponding windows of analysis through the Haar Wavelet.*

As we can see, if we assume that we are analyzing a signal on a time scale eight long then since the Haar Wavelet is only of length one, if un-dilated, there are eight distinct 'windows,' where the wavelet can be translated to, for which a coefficient from the wavelet transform can be obtained. These coefficients would in theory capture the high frequencies of the signals. Alternatively, when $j = 3$ meaning that the wavelet function is dilated by $2^3 = 8$ there is only one 'window' and so information about the signal as a whole is obtained, but without much detail and in theory would capture the lower frequency oscillations of the signal.

It is in this order of resolution that the wavelet transform is done. As we will see, the matrix operations follow this multiresolutional analysis. However, before this is evident we need to construct the wavelet transform matrices.

4. Derivations of Discrete Orthonormal Wavelet Transforms

Through the derivations we will see that some of the systems of equations turn out to be under-constrained. In order to solve this problem we say that a function ψ has M vanishing moments if $\int_{-\infty}^{\infty} \psi(x)x^m \mathrm{d}x = 0$ for $m = 0, 1, \ldots, M-1$. This automatically implies that ψ is orthogonal to polynomials of degree $M-1$.[6]

Wavelets are usually designed with vanishing moments, which make them orthogonal to low degree polynomials and so tend to compress non-oscillating functions. For our purposes we will simply say that if ψ has M vanishing moments, then the polynomials of degree $M-1$ are reproduced by the scaling functions. This is often referred to as the approximation order of the multiresolutional analysis. The constraints imposed on orthogonal compactly supported wavelets imply that if ψ has M moments then its support is at least of length $2M-1$. Thus, we see there is a trade-off between length of support and vanishing moments. However, if the function has a few singularities and is smooth between singularities, then we take advantage of the vanishing moments to solve the under-constrained systems of equations.[5]

Furthermore, the wavelet transform looks to calculate both a running average, or smoothing, of the signal as well as a specific type of running difference, or detail, of the signal. Thus, the coefficients are laid out in the matrices as shown.

4.1 The Haar Wavelet (or Daubechies-2)

We begin our derivation with the following matrix.

$$C = \begin{pmatrix}
c_0 & c_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
c_1 & -c_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_0 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_1 & -c_0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & -c_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & c_0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & c_1 & -c_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c_0 & c_1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c_1 & -c_0
\end{pmatrix}_{n \times n} \tag{4}$$

Our decision of the coefficients and their sign come from the average and difference calculation to be done. Furthermore, we see that the shifting involved between each pair of rows allows for the matrix to be orthogonal thus making this construction of the matrix appropriate. Furthermore, since we are considering the orthogonal wavelet transform we know that $C \times C^T = I$.

Thus given,

$$C^T = \begin{pmatrix} c_0 & c_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & -c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & -c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_1 & -c_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_1 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_0 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_1 & -c_0 \end{pmatrix}_{n \times n} \tag{5}$$

We see that

$$C \times C^T = \begin{pmatrix} c_0^2 + c_1^2 & 0 & \cdots & 0 & 0 \\ 0 & c_0^2 + c_1^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_0^2 + c_1^2 & 0 \\ 0 & 0 & \cdots & 0 & c_0^2 + c_1^2 \end{pmatrix}_{n \times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n \times n} \tag{6}$$

Thus, we see from the diagonal terms that

$$c_0^2 + c_1^2 = 1 \tag{7}$$

However, the off diagonal terms only yields the trivial equation $0 = 0$.

We note then that this system of equations is under-constrained. Since we are solving for two variables we will need one more equations in order to be able to solve for c_0, c_1.
We therefore apply a further restriction onto this system.

We say that the Haar wavelet transform has $M = 1$, one vanishing moment, only constant functions are reproduced by the scaling function or in other words the transform will have the detail coefficients equaling all zero and thus our second equation becomes

$$c_1 - c_0 = 0 \tag{8}$$

Though this is not a linear system of equations it is solvable through algebraic manipulation assuming positive square roots of real numbers.
So we see,

$$c_0 = \frac{1}{\sqrt{2}}, c_1 = \frac{1}{\sqrt{2}} \tag{9}$$

These yield the scaling function, father wavelet,

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{10}$$

and the wavelet function, mother wavelet,

$$\psi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

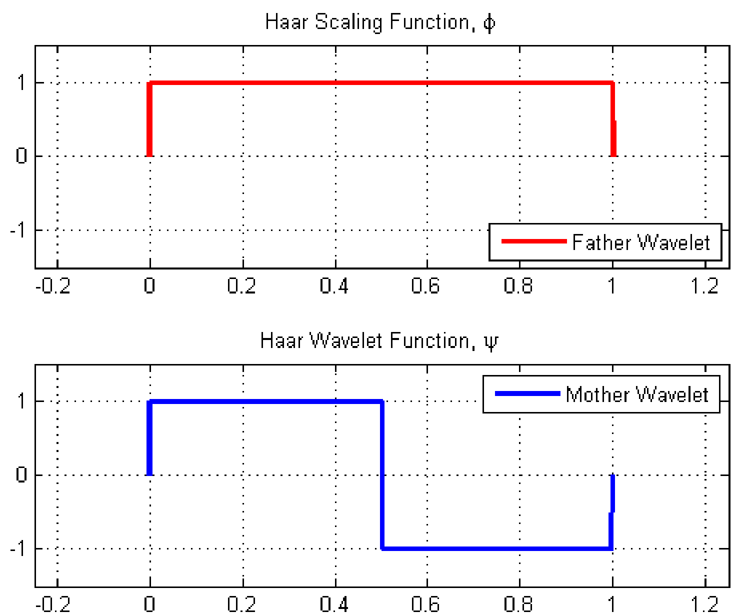

Figure 2: *Haar Wavelet's Scaling Function and Wavelet Functions plotted.*

The Haar wavelet functions are thus,

$$\Psi(x) = \begin{cases} -2^{j/2} & \text{for } 2^{-j}k \leq t < 2^{-j}(k + 1/2) \\ 2^{j/2} & \text{for } 2^{-j}(k + 1/2) \leq x < 2^{-j}(k+1) \\ 0 & \text{elsewhere} \end{cases} \tag{12}$$

4.2 The Daubechies-4 Wavelet

Here we will look the Daubechies-4 Wavelet, where we again begin our derivation of the wavelet transform with a matrix of similar form except for the last two rows which 'wrap around.'

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ c_3 & -c_2 & c_1 & -c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & c_2 & c_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & -c_2 & c_1 & -c_0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & -c_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\,\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_3 & -c_2 & c_1 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_0 & c_1 \\ c_1 & -c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_3 & -c_2 \end{pmatrix}_{n \times n} \tag{13}$$

Again, our decision of the coefficients stem from the desired average and difference calculation. Thus, since we still are considering the orthogonal wavelet transform we know once more that $C \times C^T = I$.
Thus given,

$$C^T = \begin{pmatrix} c_0 & c_3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_2 & c_1 \\ c_1 & -c_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_3 & -c_0 \\ c_2 & c_1 & c_0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ c_3 & -c_0 & c_1 & -c_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & c_1 & c_0 & c_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & -c_0 & c_1 & -c_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_3 & -c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_2 & c_1 & c_0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_3 & -c_0 & c_1 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_2 & c_1 & c_0 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_3 & -c_0 & c_1 & -c_2 \end{pmatrix}_{n\times n} \tag{14}$$

We see that

$$\begin{aligned} C \times C^T &= \begin{pmatrix} c_0^2+c_1^2+c_2^2+c_3^2 & c_0c_2+c_1c_3 & \cdots & c_0c_2+c_1c_3 & c_0c_2+c_1c_3 \\ c_0c_2+c_1c_3 & c_0^2+c_1^2+c_2^2+c_3^2 & \cdots & c_0c_2+c_1c_3 & c_0c_2+c_1c_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_0c_2+c_1c_3 & c_0c_2+c_1c_3 & \cdots & c_0^2+c_1^2+c_2^2+c_3^2 & c_0c_2+c_1c_3 \\ c_0c_2+c_1c_3 & c_0c_2+c_1c_3 & \cdots & c_0c_2+c_1c_3 & c_0^2+c_1^2+c_2^2+c_3^2 \end{pmatrix}_{n\times n} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n} \end{aligned} \tag{15}$$

Thus, we see from the diagonal elements that

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1 \tag{16}$$

and from the off diagonal elements

$$c_0c_2 + c_1c_3 = 0 \tag{17}$$

We note though that this system of equations is under-constrained. Since we are solving for four variables we will need two more equations in order to be able to solve for c_0, c_1, c_2, c_3.

We therefore apply the following restrictions on this system. We say that the Daubechies-4 wavelet transform has $M = 2$, two vanishing moments, constant functions and linear functions are reproduced by the scaling function or in other words the transform will have the detail coefficients equaling all zero and thus our third and fourth equations become

$$c_3 - c_2 + c_1 - c_0 = 0 \tag{18}$$

$$0c_3 - 1c_2 + 2c_1 - 3c_0 = 0 \tag{19}$$

Though this is not a linear system of equations it is solvable through algebraic manipulation assuming positive square roots of real numbers the Daubechies four-term scaling filters are

$$c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}} \tag{20}$$

Despite there is being no closed form equation to represent the scaling function or the wavelet function, using the matrix representation of the wavelet transform, we can graphically represent the wavelet's scaling function and wavelet function.

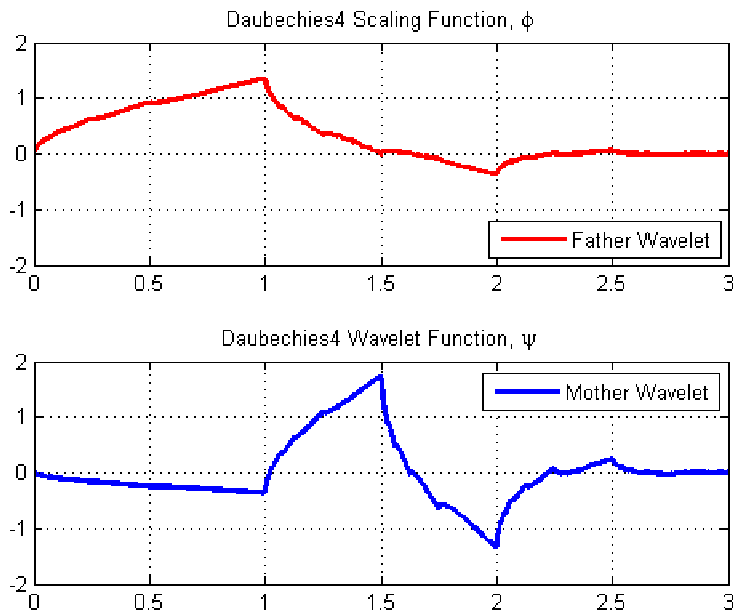

Figure 3: *Daubechies-4 Wavelet's Scaling Function and Wavelet Functions plotted.*

Note that process for deriving Daubechies wavelets with more coefficients follows the same process and only requires appropriate number of vanishing moments to solve for the inevitable under constrained system of equations obtained from the $C \times C^T = I$ computation.

Furthermore, as we will see, the wavelet functions, ψ, are what are mainly applied to the signals directly. The father wavelet, or scaling function, as it's name suggests exists to scale the mother wavelet, wavelet function, to the appropriate size for it to applied to the signal. For instance, if we again considered a signal of length eight, with the normal mother Haar wavelet of length one, there would be a maximum of eight translations when un-dilated and four dilations, as seen in Figure 1. However, if the scaling function were to be of length two, then the mother wavelet would also now be of length two and there would be a maximum of four translations when un-dilated and three dilations. Note that these maximum counts exclude dilations that 'shrink' the wavelet function since that could be done infinitely, if desired. These examples are simply meant to illustrate the notion of controlling the scale of the mother wavelet via the father wavelet.

5. Applications of Discrete Orthonormal Wavelet Transforms

If we now want to apply the discrete wavelet transform we see it is done in the following iterative way. This process is sometimes referred to as the pyramidal algorithm, for what will become obvious reasons. We will show the energized application, a symbolic application with a signal with eight elements and finally a numeric example.

5.1 General Application: Haar and Daubechies-4 Wavelet Transforms

Given some signal, $\overrightarrow{X}$, we will here for ease of calculation and representation, but without loss of generality, assume that this signal is sampled with a positive power of two number of elements. In general, the only condition on $\overrightarrow{X}$ is that it have an even number of elements.
Then, we see with C a $n \times n$ matrix and $\overrightarrow{X}$ an $n \times 1$ vector.

The Haar Wavelet transform would yield:
$C_{Haar} \times \overrightarrow{X} =$

$$\begin{pmatrix} c_0 & c_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ c_1 & -c_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & -c_0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -c_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & c_0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & c_1 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c_0 & c_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c_1 & -c_0 \end{pmatrix}_{n \times n} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_{n-5} \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix}_{n \times 1} = \begin{pmatrix} c_0 x_0 + c_1 x_1 \\ c_1 x_0 - c_0 x_1 \\ c_0 x_2 + c_1 x_3 \\ c_1 x_2 - c_0 x_3 \\ c_0 x_4 + c_1 x_5 \\ c_1 x_4 - c_0 x_5 \\ \vdots \\ c_0 x_{n-5} + c_1 x_{n-4} \\ c_1 x_{n-5} - c_0 x_{n-4} \\ c_0 x_{n-3} + c_1 x_{n-2} \\ c_1 x_{n-3} - c_0 x_{n-2} \\ c_1 x_{n-1} + c_0 x_n \\ c_0 x_{n-1} - c_1 x_n \end{pmatrix}_{n \times 1} \tag{21}$$

While the Daubechies-4 Wavelet transform would yield:

$$
C_{Daub4} \times \overrightarrow{X} = \begin{pmatrix}
c_0 & c_1 & c_2 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
c_3 & -c_2 & c_1 & -c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_0 & c_1 & c_2 & c_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_3 & -c_2 & c_1 & -c_0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_0 & c_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_3 & -c_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_0 & c_1 & c_2 & c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_3 & -c_2 & c_1 & -c_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_0 & c_1 & c_2 & c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_3 & -c_2 & c_1 & -c_0 \\
c_2 & c_3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_0 & c_1 \\
c_1 & -c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & c_3 & -c_2
\end{pmatrix}_{n \times n}
\begin{pmatrix}
x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_{n-5} \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n
\end{pmatrix}_{n \times 1}
$$

$$
= \begin{pmatrix}
c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 \\
c_3x_0 - c_2x_1 + c_1x_2 - c_0x_3 \\
c_0x_2 + c_1x_3 + c_2x_4 + c_3x_5 \\
c_3x_2 - c_2x_3 + c_1x_4 - c_0x_5 \\
c_0x_4 + c_1x_5 + c_2x_6 + c_3x_7 \\
c_3x_4 - c_2x_5 + c_1x_6 - c_0x_7 \\
\vdots \\
c_0x_{n-5} + c_1x_{n-4} + c_2x_{n-3} + c_3x_{n-2} \\
c_3x_{n-5} - c_2x_{n-4} + c_1x_{n-3} - c_0x_{n-2} \\
c_0x_{n-3} + c_1x_{n-2} + c_2x_{n-1} + c_3x_n \\
c_3x_{n-3} - c_2x_{n-2} + c_1x_{n-1} - c_0x_n \\
c_2x_0 + c_3x_1 + c_0x_{n-1} + c_1x_n \\
c_1x_0 - c_0x_1 + c_3x_{n-1} - c_2x_n
\end{pmatrix}_{n \times 1} \tag{22}
$$

As seen below, both these resulting matrices can be represented with S terms signifying the signal average, or smoothing, calculation and d terms signifying the signal difference, or detail, calculation. Furthermore, the first index of each element represents in which iteration of the transform it was computed while the second represents its position in the array at the time it was computed. Finally, the $--_i$ partition the array between different iterations of the transform with i being the total number of elements above the partition.

$$
\begin{pmatrix} S_{1,0} \\ d_{1,0} \\ S_{1,1} \\ d_{1,1} \\ S_{1,2} \\ d_{1,2} \\ \vdots \\ S_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-2} \\ S_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}-1} \\ S_{1,\frac{n}{2}} \\ d_{1,\frac{n}{2}} \end{pmatrix}_{n\times 1} \xrightarrow{sort} \begin{pmatrix} S_{1,0} \\ S_{1,1} \\ S_{1,2} \\ \vdots \\ S_{1,\frac{n}{2}-2} \\ S_{1,\frac{n}{2}-1} \\ S_{1,\frac{n}{2}} \\ --_{n/2} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ \vdots \\ d_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}} \end{pmatrix}_{n\times 1} \tag{23}
$$

This would be the end of the first permutation of the wavelet transform. We then take the S sub-array and now multiply it by a new C matrix half the size of the previous one and iterate through as shown. Note that the dimension of each necessary C matrix is noted below the arrow.

$$
\begin{pmatrix} S_{1,0} \\ S_{1,1} \\ S_{1,2} \\ \vdots \\ S_{1,\frac{n}{2}-2} \\ S_{1,\frac{n}{2}-1} \\ S_{1,\frac{n}{2}} \\ --_{n/2} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ \vdots \\ d_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}} \end{pmatrix}_{n\times 1} \xrightarrow[n/2\times n/2]{C} \begin{pmatrix} S_{2,0} \\ d_{2,0} \\ S_{2,1} \\ d_{2,1} \\ \vdots \\ S_{2,\frac{n}{4}-1} \\ d_{2,\frac{n}{4}-1} \\ S_{2,\frac{n}{4}} \\ d_{2,\frac{n}{4}} \\ --_{n/2} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ \vdots \\ d_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}} \end{pmatrix} \xrightarrow{sort} \begin{pmatrix} S_{2,0} \\ S_{2,1} \\ \vdots \\ S_{2,\frac{n}{4}-1} \\ S_{2,\frac{n}{4}} \\ --_{n/4} \\ d_{2,0} \\ d_{2,1} \\ \vdots \\ d_{2,\frac{n}{4}-1} \\ d_{2,\frac{n}{4}} \\ --_{n/2} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ \vdots \\ d_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}} \end{pmatrix} \xrightarrow[n/4\times n/4]{C} \begin{pmatrix} S_{3,0} \\ d_{3,0} \\ \vdots \\ S_{3,\frac{n}{8}} \\ d_{3,\frac{n}{8}} \\ --_{n/4} \\ d_{2,0} \\ d_{2,1} \\ \vdots \\ d_{2,\frac{n}{4}-1} \\ d_{2,\frac{n}{4}} \\ --_{n/2} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ \vdots \\ d_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}} \end{pmatrix} \xrightarrow{sort} \begin{pmatrix} S_{3,0} \\ \vdots \\ S_{3,\frac{n}{8}} \\ --_{n/8} \\ d_{3,0} \\ \vdots \\ d_{3,\frac{n}{8}} \\ --_{n/4} \\ d_{2,0} \\ d_{2,1} \\ \vdots \\ d_{2,\frac{n}{4}-1} \\ d_{2,\frac{n}{4}} \\ --_{n/2} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ \vdots \\ d_{1,\frac{n}{2}-2} \\ d_{1,\frac{n}{2}-1} \\ d_{1,\frac{n}{2}} \end{pmatrix} \xrightarrow{\cdots} \tag{24}
$$

This process can be repeated until the desired resolution is attained. Otherwise, the transform will cease at different times for each of the wavelet transforms. For the Haar wavelet Transform, when an S vector of length one is obtained the process will end since the C matrix for the Haar wavelet transform is at minimum a 2×2 and thus at least two elements are necessary in order to preform the operation. Similarly, for the Daubechies-4 Wavelet Transform, when an S vector of length two is obtained the process will end since the C matrix for the Daubechies-4 wavelet transform is at minimum a 4×4 and thus at least four elements are necessary in order to preform the operation.

5.1.1 Alternative Representation of General Application of Haar Wavelet Transform

If we want to express the Haar wavelet transform as one matrix multiplication rather than perform the pyramidal algorithm, through some work, we can see that given we know $c_0 = c_1$ if we let $h = c_0 = c_1$, then we see that the matrix

$$H = \begin{pmatrix} h^p & h^p & h^p & h^p & \cdots & \cdots & h^p & h^p & h^p & h^p \\ h^p & h^p & \cdots & h^p & h^p & -h^p & -h^p & \cdots & -h^p & -h^p \\ h^{p-1} & h^{p-1} & \cdots & -h^{p-1} & -h^{p-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & h^{p-1} & h^{p-1} & \cdots & -h^{p-1} & -h^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h & -h & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & h & -h & \cdots & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & h & -h & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & h & -h \end{pmatrix}_{2^p \times 2^p} \tag{25}$$

where p is some positive integer.

performs the entire pyramidal algorithm.

This matrix exemplifies the multiresolutional analysis feature of the Haar wavelet transform. It is clear that the first row calculates the overall average of the signal, which acts as the father wavelet, while the second row performs the difference, detail, calculation of the two halves of the signal, which acts as the first mother wavelet. Each subsequent set of rows, acting as subsequent resolutions of the mother wavelet, then perform the difference, detail, calculation on smaller partitions. This is process is continued until the last half of the rows of the H matrix perform the difference, detail, calculation at the smallest resolution.

Though it involves more computation, the pyramidal algorithm is used if a certain resolution of the Haar wavelet transform is desired since this computation is limited to only producing the result of the entire pyramidal algorithm.

5.2 8×8 Symbolic Application: Haar Wavelet Transform

We now, for clarity, show an example of the Haar wavelet transform from beginning to end using a $\overrightarrow{X}$ of length eight. Note that we could have just as easily chosen to use the Daubechies-4 and/or any other $\overrightarrow{X}$ of a power of two in length . Thus, as shown before, the transform begins with,

$$C_{8\times 8} \times \overrightarrow{X} = \begin{pmatrix} c_0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & -c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & -c_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_0 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & -c_0 \end{pmatrix}_{8\times 8} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}_{8\times 1} = \begin{pmatrix} c_0x_0 + c_1x_1 \\ c_1x_0 - c_0x_1 \\ c_0x_2 + c_1x_3 \\ c_1x_2 - c_0x_3 \\ c_0x_4 + c_1x_5 \\ c_1x_4 - c_0x_5 \\ c_0x_6 + c_1x_7 \\ c_1x_6 - c_0x_7 \end{pmatrix}_{8\times 1} \tag{26}$$

Again, now representing this in our S and d form as before we see,

$$\begin{pmatrix} S_{1,0} \\ d_{1,0} \\ S_{1,1} \\ d_{1,1} \\ S_{1,2} \\ d_{1,2} \\ S_{1,3} \\ d_{1,3} \end{pmatrix}_{8\times 1} \xrightarrow{sort} \begin{pmatrix} S_{1,0} \\ S_{1,1} \\ S_{1,2} \\ S_{1,3} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ d_{1,3} \end{pmatrix} \xrightarrow[4\times 4]{C} \begin{pmatrix} S_{2,0} \\ d_{2,0} \\ S_{2,1} \\ d_{2,1} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ d_{1,3} \end{pmatrix} \xrightarrow{sort} \begin{pmatrix} S_{2,0} \\ S_{2,1} \\ d_{2,0} \\ d_{2,1} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ d_{1,3} \end{pmatrix} \xrightarrow[2\times 2]{C} \begin{pmatrix} S_{3,0} \\ d_{3,0} \\ d_{2,0} \\ d_{2,1} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ d_{1,3} \end{pmatrix} \xrightarrow{sort} \begin{pmatrix} S_{3,0} \\ d_{3,0} \\ d_{2,0} \\ d_{2,1} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ d_{1,3} \end{pmatrix} \tag{27}$$

If we now perform this same operation using our alternative representation of the Haar wavelet transform, H, through only one matrix operation, we then see, from our notation regarding this representation above, that $p = 3$, since $8 = 2^3$, and so the Haar wavelet transform becomes,

$$H_{8\times 8} \times \overrightarrow{X} = \begin{pmatrix} h^3 & h^3 & h^3 & h^3 & h^3 & h^3 & h^3 & h^3 \\ h^3 & h^3 & h^3 & h^3 & -h^3 & -h^3 & -h^3 & -h^3 \\ h^2 & h^2 & -h^2 & -h^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h^2 & h^2 & -h^2 & -h^2 \\ h & -h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & -h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h & -h \end{pmatrix}_{8\times 8} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}_{8\times 1} = \begin{pmatrix} S_{3,0} \\ d_{3,0} \\ d_{2,0} \\ d_{2,1} \\ d_{1,0} \\ d_{1,1} \\ d_{1,2} \\ d_{1,3} \end{pmatrix}_{8\times 1} \tag{28}$$

5.3 8×8 Numerical Application: Haar Wavelet Transform

Finally, let us show a numerical example of the Haar wavelet transform with some signal with eight elements.

Let, $\overrightarrow{X}^T = [9 \quad 7 \quad 3 \quad 5 \quad 6 \quad 10 \quad 2 \quad 6]$.

Then we see,

$$
\begin{pmatrix} 9\\7\\3\\5\\6\\10\\2\\6 \end{pmatrix}_{8\times 1}
\xrightarrow[8\times 8]{C}
\begin{pmatrix}
\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}\\
\frac{9}{\sqrt{2}}-\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}-\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}-\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}-\frac{6}{\sqrt{2}}
\end{pmatrix}
\xrightarrow{sort}
\begin{pmatrix}
\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}\\
\frac{9}{\sqrt{2}}-\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}-\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}-\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}-\frac{6}{\sqrt{2}}
\end{pmatrix}
\xrightarrow[4\times 4]{C}
\begin{pmatrix}
\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}\\
\frac{9}{\sqrt{2}}-\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}-\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}-\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}-\frac{6}{\sqrt{2}}
\end{pmatrix}
\xrightarrow{sort}
\begin{pmatrix}
\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}\\
\frac{9}{\sqrt{2}}-\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}-\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}-\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}-\frac{6}{\sqrt{2}}
\end{pmatrix}
$$

$$
\xrightarrow[2\times 2]{C}
\begin{pmatrix}
\frac{\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}+\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{9}{\sqrt{2}}+\frac{7}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{3}{\sqrt{2}}+\frac{5}{\sqrt{2}}}{\sqrt{2}}\\
\frac{\frac{6}{\sqrt{2}}+\frac{10}{\sqrt{2}}}{\sqrt{2}}-\frac{\frac{2}{\sqrt{2}}+\frac{6}{\sqrt{2}}}{\sqrt{2}}\\
\frac{9}{\sqrt{2}}-\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}-\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}-\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}-\frac{6}{\sqrt{2}}
\end{pmatrix}
\xrightarrow[simplify]{sort}
\begin{pmatrix}
\frac{9}{\sqrt{8}}+\frac{7}{\sqrt{8}}+\frac{3}{\sqrt{8}}+\frac{5}{\sqrt{8}}+\frac{6}{\sqrt{8}}+\frac{10}{\sqrt{8}}+\frac{2}{\sqrt{8}}+\frac{6}{\sqrt{8}}\\
\left(\frac{9}{\sqrt{8}}+\frac{7}{\sqrt{8}}+\frac{3}{\sqrt{8}}+\frac{5}{\sqrt{8}}\right)-\left(\frac{6}{\sqrt{8}}+\frac{10}{\sqrt{8}}+\frac{2}{\sqrt{8}}+\frac{6}{\sqrt{8}}\right)\\
\left(\frac{9}{\sqrt{4}}+\frac{7}{\sqrt{4}}\right)-\left(\frac{3}{\sqrt{4}}+\frac{5}{\sqrt{4}}\right)\\
\left(\frac{6}{\sqrt{4}}+\frac{10}{\sqrt{4}}\right)-\left(\frac{2}{\sqrt{4}}+\frac{6}{\sqrt{4}}\right)\\
\frac{9}{\sqrt{2}}-\frac{7}{\sqrt{2}}\\
\frac{3}{\sqrt{2}}-\frac{5}{\sqrt{2}}\\
\frac{6}{\sqrt{2}}-\frac{10}{\sqrt{2}}\\
\frac{2}{\sqrt{2}}-\frac{6}{\sqrt{2}}
\end{pmatrix}
\tag{29}
$$

If we now perform this same operation on our sample signal $\overrightarrow{X}$ using our alternative representation of the Haar wavelet transform, H, through only one matrix operation, we see,

$$
\begin{pmatrix}
\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}}\\
\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}}\\
\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}}\\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}_{8\times 8}
\begin{pmatrix} 9\\7\\3\\5\\6\\10\\2\\6 \end{pmatrix}_{8\times 1}
=
\begin{pmatrix}
\frac{9+7+3+5+6+10+2+6}{\sqrt{8}}\\
\frac{9+7+3+5-6-10-2-6}{\sqrt{8}}\\
\frac{9+7-3-5}{\sqrt{4}}\\
\frac{6+10-2-6}{\sqrt{4}}\\
\frac{9-7}{\sqrt{2}}\\
\frac{3-5}{\sqrt{2}}\\
\frac{6-10}{\sqrt{2}}\\
\frac{2-6}{\sqrt{2}}
\end{pmatrix}_{8\times 1}
\tag{30}
$$

We note that both these results are identical confirming that through some work the construction of this alternate representation yields the same Haar wavelet transform.

6. Conclusion

In conclusion we see that the discrete wavelet transform as compared to The discrete Fourier transform for signal analysis brings some improvements. We discussed some of the disadvantages of the discrete Fourier transform and the discrete windowed Fourier transform and showed that a multiresolutional analysis based approach solves many of these problems. From the definition of a multiresolutional analysis we were able to construct a general equation for wavelet transforms and showed that the family of Daubechies wavelets, in particular Daubechies-2, Haar, wavelet and Daubechies-4 wavelet, followed the constraints of this equation and are particular implementations of a multiresolutional analysis.

Today, the wavelet transform is used in variety of ways in the signal processing field. Because of wavelet transform's ability to preserve a vast majority of the signals information even after halving or quartering the size of the signal it is often used in data compression for images and other file types. Furthermore, as a consequence of rectifying the flaw of the Fourier transform, which is not good for recognizing rapid changes in a signal, the wavelet transform is also used in today's research as a method of classification and recognition of signals that are considered irregular or chaotic. It is one of the most common methods of analyzing finger prints and is quickly becoming a popular method of analyzing medical signals which by nature become interesting when rapid changes occur.

The analysis of signals has been an important part of mathematics for some time and it has only become more important as we attempt to understand and model more complicated systems, it allows us to understand that which we previously could not. "Analysis makes them actual and measurable and seems to be a faculty of human reason meant to compensate for the brevity of life and the imperfection of our senses."-Joseph Fourier, La Théorie Analytique de la Chaleur

Acknowledgments

I would like to thanks Professor Allen Moser for his help in understanding the importance of wavelets and their applications in engineering. I would also like to thank Professor Janet Talvacchia for her invaluable guidance in writing this paper and advising me throughout the process.

References

[1] B. Burke. *A Positron Named Priscilla: Scientific Discovery at the Frontier*, chapter 7. The Mathematical Microscope: Waves, Wavelets, and Beyond. National Academy of Science, 1994.

[2] M. W. Frazier. *An Introduction to Wavelets through Linear Algebra*. Springer, 1999.

[3] E. Hernández and G. Weiss. *A First Course on Wavelets*. CRC, 1996.

[4] G. Kaiser. *A Friendly Guide to Wavelets*. Birkhäuser, 1994.

[5] M. J. Mohlenkamp and M. C. Pereyra. *Wavelets, Their Friends and What They Can Do for You*. EMS Series of Lectures in Mathematics. European Mathematical Society, 2008.

[6] J. S. Walker. *A Primer on Wavelets and their Scientific Applications*. Chapman & Hall/CRC, 1999.

[7] P. WojTaszczyk. *A Mathematical Introduction to Wavelets*. Number 37 in London Mathematical Society Student Texts. Cambridge University Press, 1997.