E19 Final Project - Expanding Modified Nodal Analysis
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For our final project we have expanded our Modified Nodal Analysis (MNA) algorithm to incorporate more circuit elements and various types of sources. In addition to the simple circuit with only dependent sources and resistors, we added more passive, linear elements, including the inductor, capacitor, and transformer. We also widened the variety of power sources from just independent sources to dependent sources and alternating current (AC) sources. We also integrated idealized versions of the operational amplifier (op amp) into our algorithm. Additionally, we implanted solution techniques for the simple circuits and single sinusoidal sources and find symbolic solutions for all other cases. Finally, we researched but did not integrate the implementation of the diodes into our algorithm.

The run file for this project is call tester.m
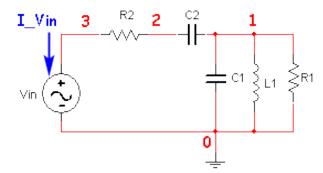
Linear Elements

1. Inductors and Capacitors

Capacitors and inductors are linear, reactive elements that only differ from resistors in that they store energy. Capacitors store voltage and inductors store current. Because capacitors and inductors are passive circuit elements like resistors, they can implement with the same approach. The key to this is using the complex impedances of inductors and capacitors as we would use resistances.

$$resistor: Z = R$$
$$capacitor: Z = \frac{1}{sC}$$
$$inductor: Z = sL$$

For parse_netlist, we list the nodes the circuit element touches just as we would for a resistor. Because we use conductance, we can just take the reciprocal of the impedance for each element.

Now we have the variable "s", which is the complex frequency. "s" is 0 for DC circuits and $j\omega$ for AC circuits.

Take the following example from Professor Cheever's website with two capacitors and one inductor.

The linear system of equations for this circuit can be derived using KCL and KVL. There are 3 nodes and 1 independent voltage source, so there will be 4 equations with 4 unknowns to generate a 4x4 matrix.

$$(v\_1)\cdot s\cdot C1 + \frac{v\_1}{s\cdot L1} + \frac{v\_1}{R1} + (v\_1 - v\_2)\cdot s\cdot C2 = 0 \qquad \text{node 1}$$

$$\frac{v\_2 - v\_3}{R2} + (v\_2 - v\_1)\cdot s\cdot C2 = 0 \qquad \text{node 2}$$

$$I\_Vin + \frac{v\_3 - v\_2}{R2} = 0 \qquad \text{node 3}$$

$$v\_3 = Vin \qquad \text{Vin}$$

Note that this system of equations is exactly what one would look like for a circuit with only resistive elements, except for the inclusion of the complex frequency 's'.

$$\mathbf{Ax = z}$$

or

$$\begin{bmatrix} s\cdot C1 + s\cdot C2 + \dfrac{1}{s\cdot L1} + \dfrac{1}{R1} & -s\cdot C2 & 0 & 0 \\ -s\cdot C2 & s\cdot C2 + \dfrac{1}{R2} & -\dfrac{1}{R2} & 0 \\ 0 & -\dfrac{1}{R2} & \dfrac{1}{R2} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v\_1 \\ v\_2 \\ v\_3 \\ I\_Vin \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Vin \end{bmatrix}$$

This is the expected result and the following shows our MNA result. There are capacitors touching nodes 1 and 2, which shows up on the diagonal of the A matrix. The inductor shows for node 1.

Netlist: projectExample1v2.cir

Vbatt 3 0 40
R2 3 2 10
C2 2 1 0.000009
C1 1 0 20
L1 1 0 30
R1 1 0 4

$$
A = \begin{bmatrix}
20s + \dfrac{0.033}{s} + 0.25 & -9 \times 10^{-6}s & 0 & 0 \\
-9 \times 10^{-6}s & 9 \times 10^{-6}s + 0.1 & -0.1 & 0 \\
0 & -0.1 & 0.1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
x = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ i_1 \end{bmatrix}
$$

$$
b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 40 \end{bmatrix}
$$

However, we should note that generally, for the netlist we switch node 1 and 3 for parsing purposes since the convention is to number nodes with sources first.

Which would change the netlist to:
Netlist: projectExample1.cir

> Vbatt 1 0 40
> R2 1 2 10
> C2 2 3 0.000009
> C1 3 0 20
> L1 3 0 30
> R1 3 0 4

Which swaps some values in the A matrix as such:

$$A = \begin{bmatrix} 0.1 & -0.1 & 0 & 1 \\ -0.1 & 9 \times 10^{-6}s + 0.1 & -9 \times 10^{-6}s & 0 \\ 0 & -9 \times 10^{-6}s & 20s + \dfrac{0.033}{s} + 0.25 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
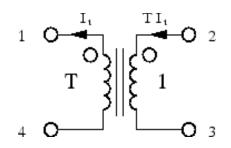
2. Transformers

In parse_netlist, we ask for two sets of sources and destinations as input. The transformer is connected to the two sets of nodes. Additionally, we specify the ratio between primary and secondary winding, which is needed when transforming the voltage and current cross nodes.

That being said, we are treating the transformer as a device similar to the function of a dependent source due to its ability to transfer voltage and current as functions another voltage and current. The only difference is that the currents/voltages being transformed have to be adjacent to each other in the actual physical circuit. But obviously this won't matter for our purpose of MNA.

In solve_netlist, since transformers are treated as power sources with a constant turns ratio between the primary and secondary windings, we do not need to apply changes to the A matrix in our system. In the B matrix, we added an additional column with entries of either '1', '-1',

'Y(j)' , '-Y(j)', or '0'. In the E matrix, we added an additional row with the following possible entries: '1', '-1', '-Y(j)', 'Y(j)' or '0', where Y(j) is the turns ratio of the jth transformer. The reason why E and B matrices are more complicated is explained using the following circuit:

We will have to add an additional equation for our additional constant I_t (which flows out of the secondary winding of the transformer):

$$Y(j)*(V2 - V3) = V1 - V4$$

And also four more for the relationship between the currents that are also induced by the transformer:

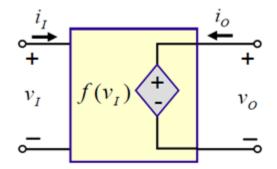$$I_1 = -I_t; \quad I_2 = T*I_t; \quad I_3 = -T*I_t; \quad I_4 = I_t$$

Thus, the augmented system of equation is as follows:

$$
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & T \\
\cdot & \cdot & \cdot & \cdot & -T \\
\cdot & \cdot & \cdot & \cdot & 1 \\
1 & -T & T & -1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
I_t
\end{bmatrix}
=
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
0
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

Sources

1. Dependent Sources

Dependent sources are voltage or current sources whose output is a function of another parameter in the circuit.

As shown in this circuit above, the voltage V0 is a function of the voltage V1. This is the case with the voltage controlled voltage source. Similar analysis can be applied to the three other kinds of dependent sources: VCCS, CCCS, and CCVS. Dependent sources only produce a voltage or current when an independent voltage or current source is in the circuit. Thus, dependent sources are treated like independent sources when using nodal analysis.

In parse_netlist, the syntax for the netlist line of the dependent source is (we will take dependent voltage sources as example):

(a) The name of the dependent source in must start with the letter D.
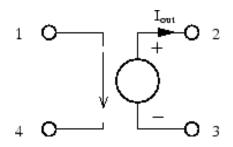(b) The first two nodes specify the nodes to which the source is connected. The first node is the positive reference; the second node is the negative reference.
(c) The second two nodes sense the voltage. The first being the positive reference, the second being the negative reference.
(d) The last thing specified is the gain of the dependent source.

In solve_netlist, since dependent sources are treated as independent sources with a constant ratio, we do not need to apply changes to the A matrix in our system. In the B matrix, we added an

additional column for the dependent source with entries of either '1' or '-1'. The rules for completing this column are the same with the independent sources. In the E matrix, we added an additional row for the dependent source with the following possible entries: '1', '-1', 'X(j)','-(j)', or '0', where X(j) is the constant ratio of the jth dependent source. The reason why E matrix is more complicated is explained using the case of a VCVS (Voltage controlled voltage source):

We will have to add an additional equation for our additional constant I_{out} (which flows out of the dependent source):

$$V_2 - V_3 = X(j)*(V_1 - V_4)$$

Thus, the augmented system of equation is as follows:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ G & -1 & 1 & -G & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_{out} \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

An example is given below:

Netlist: projectExample2.cir

Vbatt 0 1 40 0 0 0 0
R1 2 1 10 0 0 0 0
C1 2 0 0.000009 0 0 0 0
R2 3 4 20 0 0 0 0
R3 0 3 30 0 0 0 0
DC1 0 4 0 0 0 3 3
TR1 0 2 0 0 0 3 100

$$A = \begin{bmatrix} 0.1 & -0.1 & 0 & 0 & 1 & -1 & 100 \\ -0.1 & 1 \times 10^{-9}s + 0.1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0.083 & -0.05 & 0 & 0 & 1 \\ 0 & 0 & -0.05 & 0.05 & 0 & 1 & -100 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 100 & -1 & 1 & -100 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 40 \\ 0 \\ 0 \end{bmatrix}$$

2. AC sources

AC sources have an additional piece of information, namely the frequency, that have to be placed in the netlist. If an element does not have a frequency then zero is entered as a place holder.

This makes the netlist 5 elements long. To deal with this, many conditions are set up to check when there is a fifth element which then multiplies the source value by $\sin \omega t$

In solve_netlist, the voltage inputs stored in the b matrix can be put in the form of $A * \sin \omega t$.

If there is only one sinusoidal input, the ω can be substituted into the A matrix, otherwise a symbolic solution of the system is kept.

The rest of the solution remains the same as this does not affect the construction of the A matrix.

As an example we use the example, with the convention of the sources placed at the first nodes, projectExample1.cir, instead now with a sinusoidal source with $\omega = 5$. (projectExample3v2.cir is the exact example from Prof. Cheever's website with a sinusoidal source.)
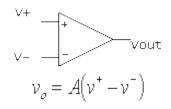
Netlist: projectExample3.cir

$$
\begin{aligned}
&\text{Vbatt 1 0 40 5} \\
&\text{R2 1 2 10 0} \\
&\text{C2 2 3 0.000009 0} \\
&\text{C1 3 0 20 0} \\
&\text{L1 3 0 30 0} \\
&\text{R1 3 0 4 0}
\end{aligned}
$$

$$
A = \begin{bmatrix}
0.1 & -0.1 & 0 & 1 \\
-0.1 & 0.1 + 4.5 \times 10^{-5}i & -4.5 \times 10^{-5}i & 0 \\
0 & -4.5 \times 10^{-5}i & 0.25 + 100i & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
x = \begin{bmatrix}
40\sin(5t) \\
(40 - 0.018i)\sin(5t) \\
(1.8 \times 10^{-5} - 3.7 \times 10^{-8}i)\sin(5t) \\
(8.1 \times 10^{-7} - 1.8 \times 10^{-3}i)\sin(5t)
\end{bmatrix}
$$

$$
b = \begin{bmatrix}
0 \\
0 \\
0 \\
40\sin(5t)
\end{bmatrix}
$$

Op Amps

Op-amps are operational amplifiers that take the difference between two input terminals and multiply by a gain to put out a greater voltage. Op-amps are mainly used with negative feedback: the output is fed back into the input to stabilize the system and reduce fluctuations. Op-amps in application have limits on their supply voltage and on their gains, but we will treat op-amps ideally here.

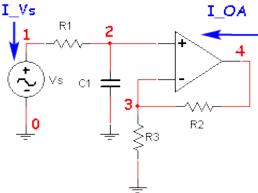$$v_o = A\left(v^+ - v^-\right)$$

Rules for op-amps under negative feedback are as follows: no current into input terminals, voltages of input terminals are equal. The key assumptions for our implementation is that the op-amp is functioning under negative feedback and that output of op-amp is treated as voltage source.

For parse_netlist, the notation goes name, positive terminal node, negative terminal node, and then output node. The op-amp is marked as a new type of element.

For solve_netlist, there are some changes in our MNA. B and C matrices are no longer transposes of each other. Instead, the B matrix treats the output node of the op-amp as a voltage source. The C matrix puts 1's for nodes touching positive terminals and -1's for nodes touching the negative terminals. While we treat the op-amp as a voltage source for the A matrix, it is not counted as a voltage source in the z matrix (independent sources).

 Take the following example from Professor Cheever's website with one capacitors and one op-amp.

As we stated earlier, we will treat the output terminal of the op-amp as a voltage source in the B matrix, so the A matrix is now 6x6 instead of just 5x5 (4 terminals and 1 independent voltage source). The new unknown variable is the current coming out of the op-amp. Note that the op-amp does not count as an independent source.

$$I\_Vs + \frac{v\_1 - v\_2}{R1} = 0 \qquad \text{node 1}$$

$$\frac{v\_2 - v\_1}{R1} + (v\_2) \cdot s \cdot C1 = 0 \qquad \text{node 2}$$

$$\frac{v\_3}{R3} + \frac{v\_3 - v\_4}{R2} = 0 \qquad \text{node 3}$$

$$I\_OA + \frac{v\_4 - v\_3}{R2} = 0 \qquad \text{node 4}$$

$$v\_1 = Vs \qquad \text{Vs}$$

$$v\_2 - v\_3 = 0 \qquad \text{OpAmp}$$

The op-amp shows up in the A matrix by stating that the voltages between its positive and negative terminal are equal, i.e. the difference between the voltages at those nodes is 0.

$$
\begin{bmatrix}
\dfrac{1}{R1} & -\dfrac{1}{R1} & 0 & 0 & 1 & 0 \\[2ex]
-\dfrac{1}{R1} & \dfrac{1}{R1}+s\cdot C1 & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & \dfrac{1}{R2}+\dfrac{1}{R3} & -\dfrac{1}{R2} & 0 & 0 \\[2ex]
0 & 0 & -\dfrac{1}{R2} & \dfrac{1}{R2} & 0 & 1 \\[2ex]
1 & 0 & 0 & 0 & 0 & 0 \\[1ex]
0 & 1 & -1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v\_1 \\ v\_2 \\ v\_3 \\ v\_4 \\ I\_Vs \\ I\_OA
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ Vs \\ 0
\end{bmatrix}
$$

We can compare this with the resulting matrix from our MNA.

Netlist: projectExample4.cir

Vs 1 0 40
R1 1 2 10
C1 2 0 0.01
OP1 2 3 4
R2 3 4 20
R3 3 0 30

$$
A = \begin{bmatrix}
0.1 & -0.1 & 0 & 0 & 1 & 0 \\
-0.1 & 0.01s + 0.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.083 & -0.05 & 0 & 0 \\
0 & 0 & -0.05 & 0.05 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
x = \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
i_1 \\
i_2
\end{bmatrix}
$$

$$
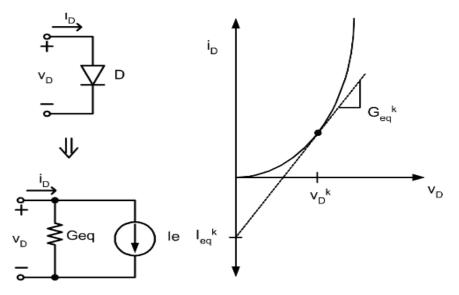b = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
40 \\
0
\end{bmatrix}
$$

Non-linear Elements

Diodes

We were unable to implement diodes in our algorithms, but we have found some research in handling the nonlinear element using the Newton-Raphson method. Diodes in general are non-linear passive circuit elements with a forward bias, meaning they only conduct current from anode to cathode. For the diode to be activated and to conduct current, the voltage across the diode must be above a certain amount: the "real" model of this current-voltage relationship is represented by the Shockley equation below with Is being the saturation current and n being the emission coefficient (two manufacturing characteristics of diodes).

$$i = I_s \left(e^{\frac{qv}{nkT}} - 1 \right)$$

A more linear model assumes a constant current once the voltage across the diode has reached a certain amount. For most diode, a constant voltage drop of 0.7V for conduction is a fair assumption. Diodes are often used for power supply management and AC-to-DC conversions. For DC circuits, the linear constant-drop model is often suitable, but for AC circuits, it is more appropriate to use the Shockley model. However, the non-linearity of this model is problematic. The Newton-Raphson approach allows us to linearize the diode and represent it as a resistor in parallel with a current source.

The left part shows the linear model of the diode while the right part shows the Shockley equation as a relation between diode voltage and current. We use Newton's method to solve for the diode voltage and then use the instantaneous slope from that point on the exponential line to find the resistance (the gain). We were unable to implement this, because the model would need to be updated each time the voltage across the diode changed (AC circuits).

See attach m-files for source codes and .cir files for netlists.

Citations:
Erik Cheever's http://www.swarthmore.edu/NatSci/echeeve1/Ref/mna/MNA1.html
Stefan Jahn's *http://qucs.sourceforge.net/tech/node16.html* for diodes
Stefan Jahn's *http://qucs.sourceforge.net/tech/node41.html* for dependent sources
Stefan Jahn's *http://qucs.sourceforge.net/tech/node48.html* for transformers