Basic Application: 1-Dimensional Malthusian Growth

We consider a population density function $\rho(x,t)$ which is modeled by

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} + r\rho, \quad r > 0 \tag{1}$$

with boundary conditions

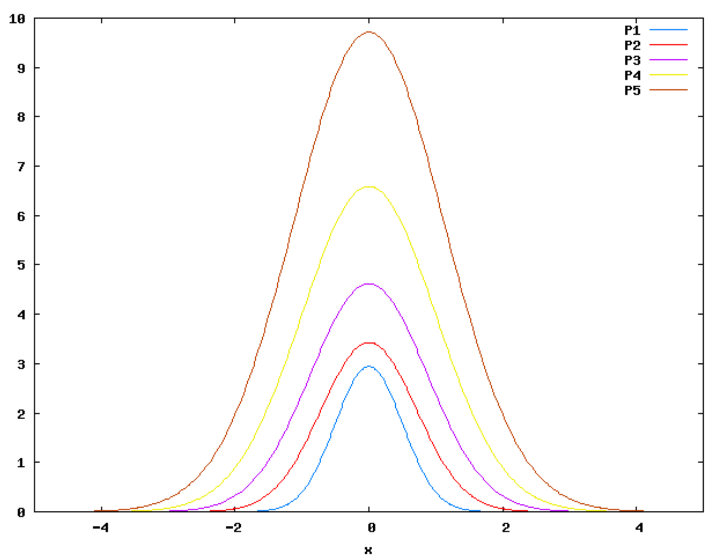
$$\rho(\infty,t) = \rho(-\infty,t) = 0, \quad \rho(x,0) = \begin{cases} \rho_0 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{2}$$

This yields the solution

$$\rho(x,t) = \frac{N}{2\sqrt{\pi D t}} e^{rt - \frac{x^2}{4Dt}} \tag{3}$$

where N is the total number of individuals at time $t = 0$. For an arbitrary $\rho(x,0)$, the solution is given by

$$\rho(x,t) = \int_{-\infty}^{\infty} \frac{S_0(x')}{2\sqrt{\pi D t}} e^{rt - \frac{(x-x')^2}{4Dt}} dx' \tag{4}$$



The speed of these waves is given by solving for $c = x/t$ and taking the asymptotic limit

$$\lim_{t\to\infty} c = \pm 2\sqrt{rD} \tag{5}$$

Next Steps: Logistic Growth

We now consider the model

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} + r\rho\left(1 - \frac{\rho}{K}\right), \quad r > 0,\, K > 0 \tag{6}$$

An exact solution for this model has not been found yet, but there have been some approximate solutions. Oddly enough, introducing a more complicated nonlinear term into the equation enables one particular form of solution. If we add the term $-2D/\rho(\partial\rho/\partial x)^2$, then we have

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} - \frac{2D}{\rho}\left(\frac{\partial \rho}{\partial x}\right)^2 + r\rho\left(1 - \frac{\rho}{K}\right) \tag{7}$$

and using the substitution $\rho = 1/G$ enables us to use a standard Fourrier method of solution to obtain

$$G(x,t) = \frac{e^{-rc}}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} G(y,0)e^{-\frac{(x-y)^2}{4Dt}}dy + r\int_0^t\int_{-\infty}^{\infty} K(y,t')^{-1}\frac{e^{-r\frac{t-t'(x-y)^2}{4D(t-t')}}}{\sqrt{4\pi D(t-t')}}dydt' \tag{8}$$

If we look for travelling wave solutions of the form $\rho(x,t) = \rho(x - ct) = S(\xi)$, then it can be shown that solutions exist only for $c \geq 2\sqrt{rD}$. The minimum velecity of propagation is equal to the ultimate speed of propagation of a Malthusian population and this speed has no dependence on the carrying capacity K!

Critical Habitat Size

From dimensional analysis, an estimate for the critical habitat size for a stable Malthusian population in a hostile environment yields $L_c = c\sqrt{D/r}$ and by detailed analysis the value of the constant is determined to be $c = \pi$. For the logistic case, things are of course more complicated, but it can be shown that as $\rho(0,t) \to 0$ then $L_c \to \pi\sqrt{D/r}$ and as $\rho(0,t) \to K$ then $L_c \to \infty$. Thus, the critical value for Malthusian growth is the minimum critical value for logistic growth. The addition of immigration terms can drastically affect the behavior of populations for which $L < L_c$, helping to prevent extinction.

Two-Dimensional Predator-Prey Systems

The spatial version of the Lotka-Volterra model (although still oversimplified) is given by

$$\frac{\partial \rho_1}{\partial t} = D_1 \frac{\partial^2 \rho_1}{\partial x^2} + a_1\rho_1 - b_1\rho_1\rho_2, \quad a_1 > 0,\, b_1 > 0 \tag{9}$$

$$\frac{\partial \rho_2}{\partial t} = D_2 \frac{\partial^2 \rho_2}{\partial x^2} - a_2\rho_2 + b_2\rho_1\rho_2, \quad a_2 > 0,\, b_2 > 0 \tag{10}$$

Stability

We examine the stability of two equilibrium points: $\rho_1 = \rho_2 = 0$ and $\rho_1 = a_2/b_2$, $\rho_2 = a_1/b_1$. For the first case, stability depends on the size of the habitat as analyzed above. For the second case, we can use a linearization argument to deduce that for a hostile environment all solutions are stable and for a reflective environment we may also obtain neutral stability. For this system, spatial fluxuations tend to die out quickly with time and time-periodic space-constant solutions dominate in the long term.

Travelling Waves

Through a complicated process, it can be shown that these equations admit travelling wave solutions of the form

$$\rho_1(\xi) \sim Ae^{-k\xi} \tag{11}$$

$$\rho_2(\xi) \sim \frac{1}{\Gamma(v+1)} \left(\frac{Ab_2}{k^2v^2}\right)^{v/2} e^{-\xi\sqrt{a_2/D_2}} \tag{12}$$

where $v = 2k^{-1}(D_2/a_2)^{1/2}$ and Γ is the gamma function.