A Simple Model's Best Hope: A Brief Introduction to Universality

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For certain classes of systems operating at a critical point, the concept of universality can offer a way out of the reductive dilemma of modeling systems with many interacting parts. In these cases, simple models can be used to classify the large scale behavior of an entire universality class. This paper provides a brief introduction to the idea of universality and the renormalization group ideas used to derive it, along with a derivation of the power law behavior of certain observables.

I. INTRODUCTION

If there is one lesson to take home from Physics 120, it is that the quantitative study of biological systems – and complex systems in general – is hard. In fact, it is extremely hard. No doubt this stems in part from the exotic and diverse behavior that these systems display, with individual problems as different from each other as whole disciplines. Yet lying underneath the teeming diversity of the individual examples is a common theme. Each of these systems arises from the countless interactions of many small parts, all operating according to a certain set of rules. We see this theme at work in systems ranging from swarms of insects to the inner workings of the cell — even in the large scale actions of evolution.

Part of the reason that these systems are so hard to study is that our normal approach to modeling breaks down. In order to accurately predict the overall behavior of the system, we would like to specify the behaviors of the individual parts as precisely as possible. However, the more precisely we specify individual behavior, the more intractable the analysis becomes when we consider the simultaneous interaction of all the constituents. This dilemma clearly poses a problem for the study of *any* complex system, and it has taken the work of some of the brightest minds of the century to even begin to overcome it in a few scattered cases.

Fortunately, it can be shown that for a small set of systems operating at a so-called *critical point*, the large-scale behavior does not necessarily depend on the precise nature of the individual parts. According to a concept from statistical physics called *universality*, systems as disparate as magnets and fluids behave almost identically near a critical point, and these behaviors in turn are almost identical to even simpler abstract models. The concept of universality can offer a way out of the modeling problem, as the overall behavior of systems belonging to the same *universality class* can be deduced from the study of extremely simple models that can often be solved exactly.

A thorough understanding of what universality is – and what it isn't – could be an invaluable tool for those embarking on the study of the types of complex systems we see around us every day. And while the language of the theory is currently couched in the mindset of statistical physics, the underlying concepts are pertinent to a much broader range of problems and really say something about the nature of modeling process itself. The purpose of this paper is to provide the quickest possible introduction to the theory behind universality using only concepts usually encountered in the study of simple population dynamics. In order to do so, some of the more complicated features of the theory must be ignored, but this does not hinder the the basic concepts. For a more complete introduction to the theory of universality, as well as an analysis of some specific examples, the reader should consult the references provided at the end of the paper.

II. THE RENORMALIZATION GROUP

Although certain experiments in statistical physics hinted at the notion of universality early in the century, it was K.G. Wilson's Nobel prize-winning application of the Renormalization Group in 1971 that finally put the concept of universality on a firm theoretical foundation. A thorough understanding of universality requires us to lay out some of the basics of this theory, which we elaborate in the following sections.

A. A "Simple" Complex System

We begin by considering a system made out of N interacting parts, where N is some large number. For the sake of simplicity, each part will be represented by a variable s which can take on the values ± 1. The interactions between these parts are modeled by a "Hamiltonian" function [1] given by

$$H = h\sum_i s_i + \sum_{i,j} K_{i,j}s_i s_j + \sum_{i,j,k} K_{i,j,k}s_i s_j s_k + \dots \quad (1)$$

where h and K represent the *coupling strength* of the interactions. In general, these coupling constants would be complex functions of the parameters of the system, such as temperature, magnetic field, etc. As a specific example, a one-dimensional system with only nearest neighbor

interactions would have a Hamiltonian given by

$$H = \sum_i K s_i s_{i+1} \,. \tag{2}$$

Once we have a Hamiltonian, we can define a "partition function"

$$Z = \mathrm{Tr}_s e^{-H} \,, \tag{3}$$

where the operation Tr_s means to sum over all the possible combinations of values for all the various s_i. This partition function is defined such that the probability of finding the system in a microstate x (i.e., $s_1 = 1$,$s_2 = -1$,...) is given by

$$P(x) = \frac{e^{-H(x)}}{Z} \,. \tag{4}$$

To complete the description of our simple system, imagine that we can define a "free-energy" function of the form

$$f = -\frac{\ln Z}{N} \tag{5}$$

and that there exists some measurable large-scale property of the system given by

$$C = \frac{\partial^2 f}{\partial T^2} \,, \tag{6}$$

where T some adjustable parameter like temperature. Such properties are common to the systems discussed in statistical physics, although their immediate application to more general systems is not always clear.

In addition to this large-scale property C, we can also describe the state of the system by means of a "correlation function" $\Gamma(\vec{r})$ which measures the tendency for any two variables separated by a "distance" $\vec{r}$ to have the same value. A simple correlation function is given by

$$\Gamma(\vec{r}) = \langle s_i s_j \rangle - \langle s \rangle^2 \,, \tag{7}$$

where the $\langle \ldots \rangle$ denote the average taken over all such variables. Since we might normally expect the correlation to decay to zero with increasing distance, we can write Γ in the form

$$\Gamma(\vec{r}) \sim r^{-\tau} e^{-r/\xi} \,, \tag{8}$$

where ξ is the *correlation length*. One definition of a critical state is one in which the correlation length becomes infinite and meaningful correlations between variables exist at all length scales. We will see how this arises from the general theory later on.

B. Renormalization Group Transformations

The renormalization group ideas are principally concerned with how the behavior of a system changes depending on the scale at which you view it. The central concept in this theory is the idea of a *scale transformation* $\mathcal{R}$ that reduces the number of variables in the system from N to N'. We can think of this transformation as a "zooming out" operation. Ordinarily, this transformation acts on the Hamiltonian function

$$H' = \mathcal{R}(H) \,. \tag{9a}$$

Alternatively, we can represent the Hamiltonian by an infinite dimensional vector in the space of all possible coupling strengths, with each component corresponding to a different type of interaction. The 1-D Hamiltonian discussed in Eq. (2) included only nearest neighbor interactions, so for this system $\vec{K} = (K_1, 0, 0, \ldots)$. When viewed this way, the transformation is defined by the relation

$$\vec{K}' = \mathcal{R}(\vec{K}) \,. \tag{9b}$$

Although this second manner of looking at the transformation function is a bit harder to visualize, it turns out to be much easier to work with mathematically.

A final constraint on the scaling transformation (in order to preserve the general form of the probabilities when viewed at differing scales) is that the partition function must be invariant:

$$Z' = \mathrm{Tr}_{s'} e^{-H'} = \mathrm{Tr}_s e^{-H} \,. \tag{10}$$

This is consistent with the idea that we are not fundamentally changing the system but are merely viewing it from a different scale.

These definitions are relatively abstract, so for a more concrete example of a scaling function, consider a 2-D system that we can represent by a grid of variables. To increase the scale of the system, we simply replace every 3×3 block of variables with a new variable s' that takes on the value $+1$ or -1 depending on whether the average of the variables in the block is positive or negative. This transformation has the effect of averaging the variables into bigger "chunks" and hence reduces the number of variables in the system from N to $N' = \frac{1}{9}N$. We can introduce the concept of the *scale factor* of the transformation, which is defined by

$$b^d = \frac{N}{N'} \,, \tag{11}$$

where d is the dimension of the system. In this case, we have $d = 2$ and so the scale factor is given by $b = 3$.

After a scaling transformation is performed, all lengths are reduced by the scaling factor according to

$$\vec{r} \Rightarrow \vec{r}' = b^{-1}\vec{r}, \tag{12}$$

because blocks that were x units away in the original scale are now only x/b units away when counting with the bigger chunks.

Finally, since the partition function remains the same, the free energy must transform as

$$f \Rightarrow f' = -\frac{\ln Z}{N'} = -b^d \frac{\ln Z}{N} \,,$$

or simply

$$f' = b^d f\,. \tag{13}$$

This will prove very useful in deriving a universal expression for the large-scale property defined earlier in Eq. (6).

C. Scale Invariance and Fixed Points

Of course, the main reason for defining the concept of a scale transformation is to classify those systems that are invariant under these transformations. Such systems are sometimes described as being *self-similar*, a word that arises often in the study of fractals and fractal-like systems that are so prevalent in the natural world. A scale-invariant system obeys the relation

$$H^* = \mathcal{R}(H^*)\,, \tag{14a}$$

which, when expressed using the coupling-space vector, yields

$$\vec{K}^* = \mathcal{R}(\vec{K}^*)\,. \tag{14b}$$

In this notation, $\vec{K}^*$ is known as a *fixed-point*, which plays roughly the same role in this theory as the concept of an equilibrium point in population dynamics.

A more visual example of a fixed point is given in Figure 1, which shows a graphical representation of the 3×3 blocking transformation discussed earlier[2]. In this pic-

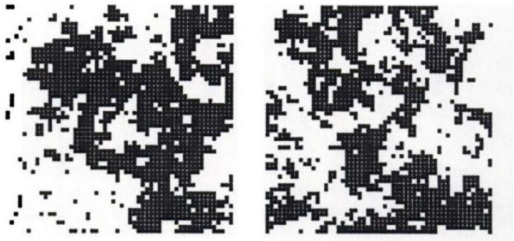

FIG. 1: A simple 2-D system near a fixed point before and after the scaling transformation.

ture, $s = \pm 1$ is represented by a black or a white square, respectively. The two images show the same system both before and after the scaling transformation. Notice that the two pictures are *statistically identical*, which is the defining feature of a fixed point. In fact, a fixed point remains statistically identical under an arbitrary number of scaling transformations, which means that meaningful structures exist at all length scales. Another way of phrasing this last statement is that for a fixed point,

$$\xi' = \xi\,. \tag{15}$$

Now, ξ has the units of length, so it must transform as $\vec{r}$ according to Eq. (12), which leads to

$$\xi = b^{-1}\xi\,, \tag{16}$$

This, in turn, implies that the correlation length must be infinite, or that correlations exist at all length scales — a property we previously claimed was a characteristic of systems at a critical point.

D. Parameter Flows

The concept of a fixed point is so powerful because as long as $\mathcal{R}$ is differentiable at $\vec{K}^*$, we can linearize the scaling transformation and expand $\vec{K}'$ as the first few terms of a power series:

$$\vec{K}' = \vec{K}^* + \mathbf{T}(\vec{K} - \vec{K}^*)\,, \tag{17a}$$

where

$$\mathbf{T} = \left.\begin{pmatrix} \frac{\partial R_1}{\partial K_1} & \frac{\partial R_1}{\partial K_2} & \cdots \\ \frac{\partial R_2}{\partial K_1} & \frac{\partial R_2}{\partial K_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right|_{\vec{K}=\vec{K}^*} \tag{17b}$$

Once in this linearized form, we can use all the tools of linear algebra to examine the behavior of the scaling transformation near the fixed point. We define the left eigenvectors $\vec{v}_i$ of $\mathbf{T}$ and their corresponding eigenvalues λ_i by the relation

$$\vec{v}_i\mathbf{T} = \lambda_i\vec{v}_i\,. \tag{18}$$

These eigenvectors will prove immensely important to us because we can write any arbitrary $\vec{K}$ as a linear combination of the eigenvectors

$$\vec{K} = \vec{K}^* + u_1\vec{v_1} + u_2\vec{v_2} + \ldots \tag{19a}$$

where the *scaling fields* u_i have been defined as

$$u_i = \vec{v_i}\cdot(\vec{K} - \vec{K}^*) = \sum_j v_{i_j}(K_j - K_j^*)\,. \tag{19b}$$

These scaling fields transform under the scaling operation as

$$\begin{aligned} u_i' &= \vec{v_i}\cdot(\vec{K}' - \vec{K}^*) = \vec{v_i}\mathbf{T}(\vec{K} - \vec{K}^*) = \lambda_i u_i \\ &= b^{y_i}u_i \end{aligned} \tag{20a}$$

where we have defined the *critical exponents* y_i such that

$$b^{y_i} = \lambda_i\,. \tag{20b}$$

Because of this transformation property, the scaling fields are classified based on the value of their critical exponent.

- $y_i > 0$: These scaling fields are *relevant* and grow increasingly larger with each scale transformation.

- $y_i < 0$: These scaling fields are *irrelevant* and grow increasingly smaller with each scale transformation.
- $y_i = 0$: These are *marginal* scaling fields and higher order terms are needed to obtain their stability behavior. We will ignore such scaling fields in the present paper.

With these transformation properties in hand, it is easy to see how an arbitrary vector $\vec{K}$ transforms under the scaling operation. Components along irrelevant eigenvectors will eventually vanish regardless of their initial starting condition, and the system will tend towards the fixed point. On the other hand, the components of the relevant eigenvectors will increase and push the system away from the fixed point.

From this behavior, we can completely characterize the path of any system undergoing successive scale transformations in the neighborhood of $\vec{K}^*$. For simplicity, we consider a fixed point in a two-dimensional parameter space containing only one relevant and one irrelevant eigenvector (see Figure 2). The eigenvectors themselves

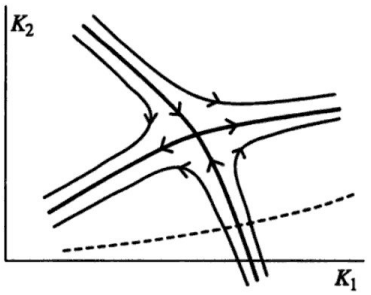

FIG. 2: Scaling trajectories near a fixed point

act as barriers that a system cannot cross. The approximate path of a system after many scaling transformations can be inferred by the directions of the arrows, and a few sample paths have been plotted in the figure.

Notice that unless the initial component of the relevant eigenvector is zero, successive scaling transformations will eventually carry the system far from the fixed point. However, if the relevant scaling fields are initially zero, then successive transformations will eventually bring the system to the fixed point, regardless of the initial conditions of the irrelevant variables. We have finally stumbled upon the true notion of universality! Every vector $\vec{K}$ along this line (which in infinite dimensions is a complex surface) represents a different model with a different Hamiltonian that models different interactions. Yet when we are only concerned with the large scale behavior of such systems (i.e., after many scaling transformations), this myriad of models is explained by considering the behavior only at the fixed point. Models that are controlled by the same fixed point in this way are said to belong to the same *universality class*.

We are now finally able to give an adequate definition of a system at a *critical point*, a term which has been particularly vague up to this point. A system at a critical point is simply one in which all the relevant scaling fields are set to zero. Now, we have already shown that the correlation length can only decrease with successive scale transformations and that the correlation length approaches infinity at the fixed point. Thus it must also be the case that $\xi \to \infty$ at a critical point as well, because all such systems much eventually reach the fixed point after a large number of scale transformations.

III. UNIVERSAL POWER LAWS

For systems near a critical point, we can often find an expression for many large-scale properties of the system in a particularly simple power law for. We will derive an example of this power law behavior for the "specific heat" property we discussed earlier, although we must make a slight modification to our original model. Suppose that the coupling strengths in the Hamiltonian were actually functions of some tunable parameter like temperature. Then one can imagine that the experimenter is able to slowly change this parameter and the system traces out a path in coupling space not unlike the dashed line in Figure 2. At exactly the critical point where the dashed line intersects the critical surface, the temperature is some critical value T_C. We define a new variable t such that

$$t = \frac{T - T_C}{T_C} \,. \tag{21}$$

For any system sufficiently close to the fixed point, we can then apply n scaling transformations to the system where n is chosen to satisfy

$$b^{nd}|t| = 1 \,, \tag{22}$$

Viewing the free energy as a function of the scaling fields u_i, after n scaling transformations we must have

$$f(u_1, u_2, \ldots) = b^{-nd} f(b^{ny_1} u_1, b^{ny_2} u_2, \ldots) \,.$$

If we further assume that we have only one relevant scaling field u_1 and that we are close enough to the fixed point that we can expand this scaling field as a series in t, then we obtain

$$f(t, u_2, \ldots) = b^{-nd} f(b^{ny_1} t, 0, \ldots) \,. \tag{23}$$

Then from the definition of our "specific heat" property, we have

$$C \sim \left(\frac{\partial^2 f}{\partial t^2} \right) = b^{-nd+2ny_1} f_{tt}(b^{ny_1} t, \ldots) \,,$$

and using Eq. (22) and a little algebra finally results in

$$C \sim |t|^{(d-2y_1)/y_1} f_{tt}(\pm 1, \ldots)$$

Since the last term is just a constant, we can simply write

$$C \sim |t|^{(d-2y_1)/y_1} \tag{24}$$

Notice that the exponent in the expression for C is a function only of the dimension of the model and the relevant critical exponent. Thus, this same power law relation is obeyed by *every* model in the universality class, despite the fact that the two systems include different interaction terms. This power law form for many of the system's measurable properties is one of the most important features of a system at a critical point, which helps explain why searching for power law forms is so popular in the current literature. However, we must be careful to remember that power laws arise because the system is at a critical point and not vice versa. As the previous sections show, being at a critical point implies much more than power law behavior. Indeed, the power laws are in many ways simply an added bonus to be found after establishing the more powerful notion of a universality class.

IV. CONCLUSION

We have seen that for systems operating at a critical point, the concept of universality implies that the large-scale behavior of the system is independent of the particular nature of the interaction terms in the model. Thus, universality offers a way out of the dilemma encountered in modeling many complex systems. While we introduced the theory for a very simple example, the ideas of the renormalization group are very powerful and can be extended to a much broader range of phenomena than we mention here. This is not to say that universality is a panacea for all complex modeling. In fact, we have seen that only a very particular class of systems obeys the necessary conditions to display universality. But for those that do, the ideas of the renormalization group provide a remarkably powerful analysis of the underlying behavior. The application of these ideas outside of the realm of statistical physics is relatively underdeveloped, but with increasing attention, universality has the potential for making remarkable progress in the field.

V. REFERENCES

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[1] If many of these terms sound like they are coming from a physics textbook, it is because they do. In this case, we do not need to know the exact physical definition of a Hamiltonian but only that it serves to completely specify the rules that govern the system.

[2] This image and the figure that follows were shamelessly taken from John Cardy's excellent *Scaling and Renormalization in Statistical Physics*, which provided much of the inspiration for this work.