Theoretical:

$$\begin{aligned}
X_t &= \int_{-\pi}^{\pi} e^{i\lambda t} dZ(\lambda) \\
c_r &= E(X_t X_{t-r}) = \int_{-\pi}^{\pi} \exp(ir\omega) f(\omega) d\omega \\
f(\omega) &= \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} c_r \exp(-ir\omega) \\
E(dZ(\lambda)\overline{dZ(\lambda)}) &= f(\lambda) \\
E(dZ(\lambda)\overline{dZ(\omega)}) &= 0
\end{aligned}$$

Empirical:

$$\begin{aligned}
\omega_j &= \frac{2\pi j}{n} \\
J_j &= \frac{1}{n} \sum_{t=0}^{n-1} X_t \exp(-i\omega_j t) \\
X_t &= \sum_{j=0}^{n-1} J_j \exp(i\omega_j t) \\
I_j &= \frac{n}{2\pi} |J_j|^2 = \frac{n}{2\pi} J_j \overline{J_j} \\
I_j &= \sum_{|r|<n} \hat{c}_r \exp(-ir\omega_j) \\
I_j/f(\omega_j) & \quad Exponential(1) \\
\hat{c}_r &= \frac{1}{n} \sum_{t=|r|}^{n-1} X_t X_{t-|r|} \\
\hat{c}_r &= \int_{-\pi}^{\pi} I(\omega) \exp(ir\omega) d\omega = \frac{2\pi}{2n} \sum_{j=0}^{2n-1} I(\omega'_j) \exp(ir\omega'_j)
\end{aligned}$$

Fourier Transform of the Boxcar: $J(\omega) = \frac{1}{n}\sum_{t=0}^{n-1} \exp(-i\omega t) = \exp(-i(n-1)\omega/2)\frac{sin(n\omega/2)}{n\sin(\omega/2)}$. Dirichlet Kernel: $D_n(\omega) = \frac{sin(n\omega/2)}{n\sin(\omega/2)}$

Transfer Functions:

$$\begin{aligned}
G(\omega) &= \sum_{u=r}^{s} g_u \exp(-i\omega u) \\
f_Z(\omega) &= |G(\omega)|^2 f_Y(\omega) \\
J_Z(\omega) &= G(\omega) J_Y(\omega)
\end{aligned}$$

Matching condition: $\int |G(\omega)|^2 dF_X(\omega) < \infty$

$nVar(\overline{X}) \to 2\pi f(0)$

$\sigma^2 = 2\pi \exp(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(f_X(\lambda))d\lambda$ (there is a spectral factorization if this is nonzero)

ARMA:

$$\begin{aligned} X_t + b_1 X_{t-1} + ... + b_p X_{t-p} &= \epsilon_t + a_1 \epsilon_{t-1} + ... + a_q \epsilon_{t-q} \\ dZ_X(\lambda) &= \frac{\sum a_j \exp(-i\lambda j)}{\sum b_k \exp(-i\lambda k)} dZ_\epsilon(\lambda) \\ f_X(\lambda) &= |\frac{\sum a_j \exp(-i\lambda j)}{\sum b_k \exp(-i\lambda k)}|^2 \frac{\sigma^2}{2\pi} \end{aligned}$$

Long Memory:

$$\begin{aligned} f(\lambda) & \quad k\lambda^{2d} \\ c_r & \quad k|r|^{2d-1} \\ Var(\overline{X_n}) & \quad k_1 n^{2d-1} \\ \binom{d}{j} &= \frac{1}{j!} d(d-1)...(d-j+1) = (-1)^j \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(d)} \\ \Delta^d &= (1-B)^d = \sum_{j=0}^{\infty} (-1)^j \binom{d}{j} B^j \end{aligned}$$

$\log I_j \approx Constant - 2d \log|\omega_j| + \epsilon_j$

GARCH:

$$\begin{aligned} \epsilon_t | \Psi_{t-1} &\sim Normal(0, h_t) \\ h_t &= \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \\ \sum \alpha_i + \sum \beta_j &< 1 \\ Var(\hat{\rho}_r) &\approx \frac{1}{n}(1 + Cov(\epsilon_t^2, \epsilon_{t-r}^2)(1 - \sum_{i=1}^{q} \alpha_i^2)/\omega^2) \end{aligned}$$