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Physics 112

Plasma Waves

Dispersion Relations

Waves of different frequency travel at different speeds in a dispersive medium, and the frequency ω can be expressed as a function of the wave number k. This is known as the *dispersion relation.* For example, light waves in a vacuum have the dispersion relation $\omega = ck$, and sound waves also have a linear dispersion relation.

In plasmas, there are a variety of waves which propagate with a variety of nonlinear dispersion relations. We will examine a few of these waves and the approximations needed to find their dispersion relations. For more information, see Goldston and Rutherford's *Plasma Physics.*

Plasma Oscillation: A Simple Model

Consider a section of plasma of cross-section A in which the electrons have been diplaced from their equilibrium condition by a distance δ, leaving the ions unmoved. The electric field in the plasma is like the field inside a parallel plate capacitor,

$$E = \frac{\sigma}{\epsilon_0} = \frac{e n_e \delta}{\epsilon_0} \tag{1}$$

where e is the electron charge and n_e is the number density of electrons. The force on the electrons due to this field is given by

$$F = Q_{total} E = (e n_e \delta A) \frac{e n_e \delta}{\epsilon_0} = \frac{e^2 n_e^2 \delta A}{\epsilon_0} \delta \tag{2}$$

By Newton's Second Law, we can also write the force as

$$F = M_{total} \ddot{\delta} = m_e n_e \delta A \ddot{\delta} \tag{3}$$

Combining Eqs. (2) and (3) gives simple harmonic motion:

$$\ddot{\delta} = \omega_p^2 \delta, \quad \omega_p = \left(\frac{n_e e^2}{m_e \epsilon_0} \right)^{\frac{1}{2}} \tag{4}$$

The electric field pulls the electrons back towards equilibrium, where they exactly neutralize the ion charge, but the kinetic energy gained in this process causes the electrons to overshoot to a new displacement on the other side.

Langmuir Waves

We will now perform a more sophisticated analysis of the same situation. Adding a term for the scalar electron pressure ($\bigtriangledown p_e$) to the net force and performing the chain rule to find $\frac{d\mathbf{u}_e}{dt}$, where $\mathbf{u}_e$ is the velocity, Newton's Second Law becomes

$$m_e n_e \left[\dot{\mathbf{u}}_e + (\mathbf{u}_e \cdot \bigtriangledown) \mathbf{u}_e \right] = -e n_e \mathbf{E} - \bigtriangledown p_e \tag{5}$$

We make the assumption that $\mathbf{u}_e$, n_e, $\mathbf{E}$, and p_e each have a constant part (like $\mathbf{u}_0$) added to a very small oscillating part (like $\mathbf{u}_1$). If the oscillations are assumed to be harmonic and in the x direction (so that the oscillating parts have the form $e^{i(kx-\omega t)}$), the $\bigtriangledown$ operator becomes ik, and $\frac{\partial}{\partial t}$ becomes $-i\omega$. Letting $u_0 = 0$, $E_0 = 0$, and neglecting second-order terms, Eq. (5) becomes

$$-i\omega m_e n_0 u_1 = -e n_0 E_1 - ik p_1 \tag{6}$$

We can write the continuity equation (statement of local electron charge conservation) as

$$\dot{n}_e + \bigtriangledown \cdot (n_e \mathbf{u}_e) = 0 \Rightarrow -i\omega n_1 + ik n_0 u_1 = 0 \tag{7}$$

To write Gauss's Law, we also have to consider the ions, since they contribute to the E-field. Assuming they each have charge $+e$ and that the ion density is n_i, Gauss's Law becomes

$$\epsilon_0 \bigtriangledown \cdot \mathbf{E} = e(n_i - n_e) \Rightarrow ik\epsilon_0 E_1 = -e n_1 \tag{8}$$

Substituting from Eqs. (7) and (8) into Eq. (6) gives

$$\frac{i\omega^2 m_e n_1}{k} = \frac{-e^2 n_0 n_1}{ik\epsilon_0} + ikp_1 \tag{9}$$

With the assumption that the electron compression occurs one-dimensionally and faster than thermal conduction, we have $p_1 = \gamma T n_1$, with $\gamma = 3$. Substituting this into Eq. (9) and rearranging, we have the Bohm-Gross dispersion relation for electrostatic plasma waves (Langmuir waves),

$$\omega^2 = \frac{e^2 n_0}{m_e \epsilon_0} + \frac{3k^2 T}{m_e} = \omega_p^2 + 3kv_{t,e}^2 \tag{10}$$

where $v_{t,e}$ is the electron thermal velocity. We can trace this additional frequency component to the inclusion of electron pressure in our forces. Note that because of it, $\omega \leq \omega_p$. Also, for long wavelengths (low k) or low temperature, the wave phase velocity $\frac{\omega}{k}$ can become larger than $v_{t,e}$, or even larger than c (which is fine, since energy travels at the group velocity, not the wave velocity). See attached Figure (1) for a graph of this dispersion relation.

Note that we can confirm that this situation is electrostatic by finding the first-order expression for $\mathbf{J}$,

$$\mathbf{J} = -en_0 u_1 = -e\frac{\omega}{k} n_1 = i\omega\epsilon_0 E_1 = -\epsilon_0 \dot{E}_1 \tag{11}$$

and seeing that Ampère's law with Maxwell's correction goes to zero:

$$\bigtriangledown \times \mathbf{B}_1 = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dot{\mathbf{E}}_1 = 0 \tag{12}$$

Ion Sound Waves

Another electrostatic plasma wave arises from the motion of the ions, caused by a pressure of the form nkT. The dispersion relation for these waves is shown in Figure (2). For small k, it has the linear form of a normal sound wave, with a slope of the ion sound speed $C_s = (T_e/M)^{1/2}$. Compare the graph of this dispersion relation with the Langmuir wave dispersion relation.

The Dielectric Tensor

More plasma waves arise when we consider a background magnetic field, $\mathbf{B}_0$. When the waves propagate perpendicular to the magnetic field ($\mathbf{k} \perp \mathbf{B}_0$), we see ordinary waves (O-waves) whose electric fields are oriented along the magnetic field ($\mathbf{E}_1 \parallel \mathbf{B}_0$) and extraordinary waves (X-waves) with $\mathbf{E}_1 \perp \mathbf{B}_0$. When the waves propagate along the magnetic field ($\mathbf{k} \parallel \mathbf{B}_0$) and when $\mathbf{E}_1 \parallel \mathbf{B}_0$, we see the Langmuir waves described earlier, but when $\mathbf{E}_1 \perp \mathbf{B}_0$, we have left and right circularly polarized waves (L and R waves).

Using tensor notation, all of these waves can be consider together. The fluid equation of motion (making the same approximation as before that each quantity has a constant part and a very small oscillating part) can be written as

$$mn_0 \frac{\partial \mathbf{u}_1}{\partial t} = qn_0 \left(\mathbf{E}_1 + \mathbf{u}_1 \times \mathbf{B}_0\right) - \gamma T \bigtriangledown n_1 \tag{13}$$

(Compare to Eq. (5).) Making the same approximation of harmonic oscillations in each dimension, and again linearizing the continuity equation, we obtain a set of linear equations for the components of $\mathbf{u}_1$. We then use Ohm's law to find the electrical conductivity tensor:

$$\mathbf{J}_1 = \sum n_0 q \mathbf{u}_1 = \underline{\sigma} \cdot \mathbf{E}_1 \tag{14}$$

The tensor conductivity can be substituted into the wave equation, and we derive

$$\left(\omega^2 \mu_0 \underline{\epsilon} - k^2 \mathbb{X}\right) \cdot \mathbf{E}_1 = 0 \tag{15}$$

where $\underline{\epsilon}$ is the dielectric tensor, given by

$$\underline{\epsilon} = \epsilon_0 \left(\mathbb{I} + \frac{i\underline{\sigma}}{\epsilon_0 \omega}\right) \tag{16}$$

and $\mathbb{X}$ is the tensor defined by $\mathbb{X} = \mathbb{I} - \mathbf{k}\mathbf{k}/k^2$, which, letting θ be the angle between $\mathbf{k}$ and $\mathbf{B}_0$ and choosing $k_y = 0$ so that $\mathbf{k} = k \sin\theta \hat{x} + k \cos\theta \hat{z}$, looks like

$$\mathbb{X} = \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix} \tag{17}$$

The dispersion relation can be found by setting the determinant of the tensor quantity in parenthesis in Eq. (15) equal to zero.

Magnetohydrodynamic Oscillation

According to *Alfvén's Theorem* (remember my presentation two weeks ago?), the magnetic flux through any closed loop moving with a plasma is constant in time, which means that the plasma is dragged around with the field lines. If some perturbation, such as a current loop, causes some field lines to be pulled together, this generates a pressure of the form $\frac{B^2}{2\mu_0}$, and the movement of the plasma to relieve this pressure results in another oscillation - Alfvén waves. We can find the characteristic speed of these waves by setting the energy density $\frac{1}{2}\rho v^2$ equal to the pressure density $\frac{B^2}{2\mu_0}$ to get

$$v_A = \left(\frac{B^2}{\rho\mu_0}\right)^{\frac{1}{2}} \tag{18}$$