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Physics 115 - Final Presentation

Vacuum Fluctuations and the Casimir Force

> I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that is something new. I told him that I was puzzled by the extremely simple form of the expressions for the interaction at very large distances and he mumbled something about zero-point energy. That was all, but it put me on a new track.
>
> —H. B. G. Casimir [1]

The Casimir Force was predicted in 1948 by Dutch physicist Hendrick Casimir. Casimir realized that when calculating the energy between two parallel uncharged conducting plates, only those virtual photons whose wavelengths fit an integral number of times into the gap should be counted. Each mode contributes to a pressure on the plates, and the infinite number of modes outside the plates is in some sense greater than the infinite number inside the plates, resulting in a small force drawing the plates together. This experimentally-confirmed force is one observable consequence of the existence of the vacuum electromagnetic field.

Theoretical Background

First, let us review the tools we have developed through Loudon to analyze the quantum vacuum. In §4.4, when he quantized the electromagnetic field, Loudon expressed the radiation Hamiltonian as

$$\hat{\mathcal{H}}_R = \sum_{\mathbf{k}} \sum_{\lambda} \hbar\omega_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k}\lambda} \hat{a}_{\mathbf{k}\lambda} + \frac{1}{2} \right). \tag{1}$$

Then in §6.2, he expressed this in continuous variables,

$$\hat{\mathcal{H}}_R = \int_0^\infty \hbar\omega \hat{a}^\dagger(\omega) \hat{a}(\omega) d\omega + \text{vacuum energy}, \tag{2}$$

where he ignored the infinite contribution due to the vacuum energy. Up till now, we have been able to ignore this infinite ground state contribution because we have been interested in measuring the intensity of a light beam, which means that we were detecting changes above this level [2]. Now we will consider the vacuum state of the electromagnetic field, in which there are no photons excited in any mode.

In §4.4, Loudon defined the vacuum state, $|\{0\}\rangle$, as the the state with no photons in any mode ($n_{\mathbf{k}\lambda} = 0$ for all $\mathbf{k}$ and λ), which means that the destruction operator gives $\hat{a}_{\mathbf{k}\lambda} |\{0\}\rangle = 0$ for all $\mathbf{k}$ and λ. Using Eq. (1), the energy eigenvalue equation becomes

$$\frac{1}{2} \sum_{\mathbf{k}} \sum_{\lambda} \hbar\omega_{\mathbf{k}} |\{0\}\rangle = \mathcal{E}_0 |\{0\}\rangle, \tag{3}$$

resulting in a vacuum energy

$$\mathcal{E}_0 = \frac{1}{2} \sum_{\mathbf{k}} \sum_{\lambda} \hbar\omega_{\mathbf{k}} = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}, \tag{4}$$

where we have summed over two polarizations.

First, consider a one-dimensional system where two conducting reflecting mirrors are placed a distance L apart. The presence of the cavity allows only discrete modes, and in §1.10 Loudon showed that boundary conditions in such a cavity require a density of modes $k = \nu\pi/L$. We can then write the energy inside the cavity using the one-dimensional version of Eq. (4):

$$\mathcal{E}_{\text{cav}} = \sum_k \hbar ck = \frac{\pi\hbar c}{L}\sum_{\nu=1}^{\infty}\nu. \tag{5}$$

The vacuum energy in the same space but without the mirrors is the same expression, with the discrete ν replaced by a continuous variable:

$$\mathcal{E}_{\text{free}} = \frac{\pi\hbar c}{L}\int_0^{\infty}\nu d\nu. \tag{6}$$

Note that both of these energies are infinite. Now consider their difference, which is the change in energy produced by the presence of the cavity:

$$\Delta\mathcal{E} = \mathcal{E}_{\text{cav}} - \mathcal{E}_{\text{free}} = \frac{\pi\hbar c}{L}\left[\sum_{\nu=1}^{\infty}\nu - \int_0^{\infty}\nu d\nu\right]. \tag{7}$$

As I will demonstrate, this can be solved by using the Euler-Maclaurin summation formula [1] and a conversion factor, $\lim_{\epsilon\to\infty} e^{-\epsilon\nu}$, resulting in

$$\Delta\mathcal{E} = -\frac{\pi\hbar c}{12L}. \tag{8}$$

There is therefore an attractive force between the two mirrors:

$$\mathcal{F} = \frac{\partial\Delta\mathcal{E}}{\partial L} = -\frac{\pi\hbar c}{12L^2}. \tag{9}$$

We thus have the fascinating result that there can be finite changes in the infinite electromagnetic vacuum energy, at least in this one-dimensional model.

Since we live in a 3D world, the modes with wavevectors that are not perpendicular to the mirrors must also be included in our analysis. If we consider a box with two sides (x and y) of length D, and the third (z) of length L, where $L \ll D$, the sums for x and y can be replaced by integrals, and the energy difference can be written by:

$$\Delta\mathcal{E} = \frac{D^2\hbar c}{\pi^2}\left[\sum_\nu \int_0^{\infty} dk_x \int_0^{\infty} dk_y \left(k_x^2 + k_y^2 + \frac{\nu^2\pi^2}{L^2}\right)^{1/2} - \frac{L}{\pi}\int_0^{\infty} dk_x \int_0^{\infty} dk_y \int_0^{\infty} dk_z \left(k_x^2 + k_y^2 + k_z^2\right)^{1/2}\right] \tag{10}$$

This time using the third derivative in the Euler-Maclaurin summation formula, we can write [1]

$$\Delta\mathcal{E} = -\left(\frac{\pi^2\hbar c}{720L^3}\right)D^2, \tag{11}$$

resulting in a force per unit area (pressure) of

$$\mathcal{P} = -\frac{\pi^2 \hbar c}{240 L^4} = \frac{0.013}{L^4} \quad \text{dynes/cm}^2, \tag{12}$$

where L is measured in μm. To get a sense of magnitude, when $L = 1\mu$m, the Coulomb force between the plates is greater than the Casimir force if there is a potential difference of only 17 mV. Note that the result of Eq. (12), while more realistic than Eq. (9), assumes that the material is perfectly reflective at all frequencies. Besides the finite conductivity of actual plates, the other important correction to the Casimir force is the effect of finite temperature [3]. To derive a corrected result is beyond the scope of this presentation; if interested, see Milonni, Chapter 6, for more detail [1].

Experiments

To measure the Casimir force between dielectrics, it is necessary to precisely measure the separation of two dielectrics as well as the force between them. Since both of these quantities are quite small, this is no easy matter.

The first attempt to measure the Casimir force between conducting plates was made by Sparnaay in 1958. He measured the force using the deflection of a spring attached to a steel beam, which was attached to a capacitor; a deflection in the spring resulted in a measurable change in the capacitor's capacitance [1]. While his results were consistent with Casimir's theory, the uncertainty was about 100% [3].

The first really accurate measurements of the Casimir force were performed by Lamoreaux and published in 1997. Lamoreaux measured the force between a flat plate and a spherical lens, making the necessary adjustments to the theoretical prediction. The plate and lens were coated with copper and gold on the faces that were brought together. The plate was connected to a torsion pendulum which measured the force between the two surfaces. The separation of the surfaces was controlled by a micropositioning assembly, and at each separation, the voltage was measured that was needed to keep the pendulum at a fixed angle. Lamoreaux measured the attractive force to within 5% of theory [3].

Later that year, Mohideen and Roy published another measurement of the Casimir force, this time to within 1% of theory. They used aluminum-coated materials, a small sphere ($200 \pm 4\mu$m diameter) on the tip of a cantilever and a flat plate. The force on the sphere was determined by measuring the deflection of the cantilever a laser [4].

Just this year, a group from Bell laboratories performed the first measurement of the Casimir force on a mechanical system. They demonstrated that when surfaces are within 100 nm, the oscillatory behavior of microstructures changes. An alternating field was applied to a metallic paddle, causing it to oscillate. They they lowered a gold-plated sphere with a diameter of 100 μm towards the paddle, and they detected changes in the amplitude and frequency of the oscillations. In fact, the Casimir force introduces various nonlinear effects, such as hysteresis [5]. Other researchers, such as Mohideen, are exploring applications of these findings to the design of micromachines, such as using a version of this oscillator as a precise position sensor [6].

Conclusions

The vacuum field is a quantum phenomenon with no classical analog. Besides the Casimir force, it is evident in the Lamb shift, spontaneous emission, van der Waals forces, and many other effects. The Casimir-Polder force, which is related to the Casimir force, describes the attraction between a conducting plate and a neutral atom [1].

It is worth noting that the attractive Casimir force between two plates that we have been considering depends on the geometry of the plates. Initially, Casimir developed a model for the electron as a spherical shell of charge, and he suggested that the Casimir force might counter the electrostatic repulsion [1]. In 1968, however, Timothy Boyer showed that the geometry of two hemispheres causes a repulsive force due to the zero-point energy, not an attractive one [7].

Besides being used in various engineering applications, the Casimir force may have important implications for theoretical physics. For instance, one (not necessarily reliable) source suggests that the Casimir force shows that if supersymmetry exists, it must be a broken symmetry, since otherwise there would be fermionic photinos whose contribution would exactly cancel that of the photons. It also claims that the vacuum energy ought to act gravitationally to produce a large cosmological constant which would cause space-time to curl up, and quantum gravity might solve this paradox [8].

References

[1] P. W. Milonni. *The Quantum Vacuum* (Academic Press, New York, 1994).

[2] R. Loudon. *The Quantum Theory of Light* (Oxford University Press, New York, 2000).

[3] S. K. Lamoreaux, Demonstration of the Casimir force in the 0.6 to 6 μm range, *Phys. Rev. Lett.* **78**, 5-8 (1997).

[4] U. Mohideen and A. Roy, Precision measurement of the Casimir force from 0.1 to 0.9 μm, *Phys. Rev. Lett.* **81**, 4549-52 (1998).

[5] H. B. Chan, *et al*, Nonlinear micromechanical Casimir oscillator, *Phys. Rev. Lett.* **87**, 211801 (2001).

[6] G. Brumfiel, Casimir force holds empty promise, *Phys. Rev. Focus*, 2 Nov. 2001. Retrieved 3 Dec. 2001 from APS website: http://focus.aps.org/v8/st25.html

[7] T. H. Boyer, Quantum electromagnetic zero-point energy of a conducting spherical shell and the Casimir model for a charged particle, *Phys. Rev.*, **174**, 1764-1776 (1968).

[8] P Gibbs, What is the Casimir effect? Retrieved 5 Dec. 2001 from Iowa State University website: http://www.public.iastate.edu/p̃hysics/sci.physics/faq/casimir.html